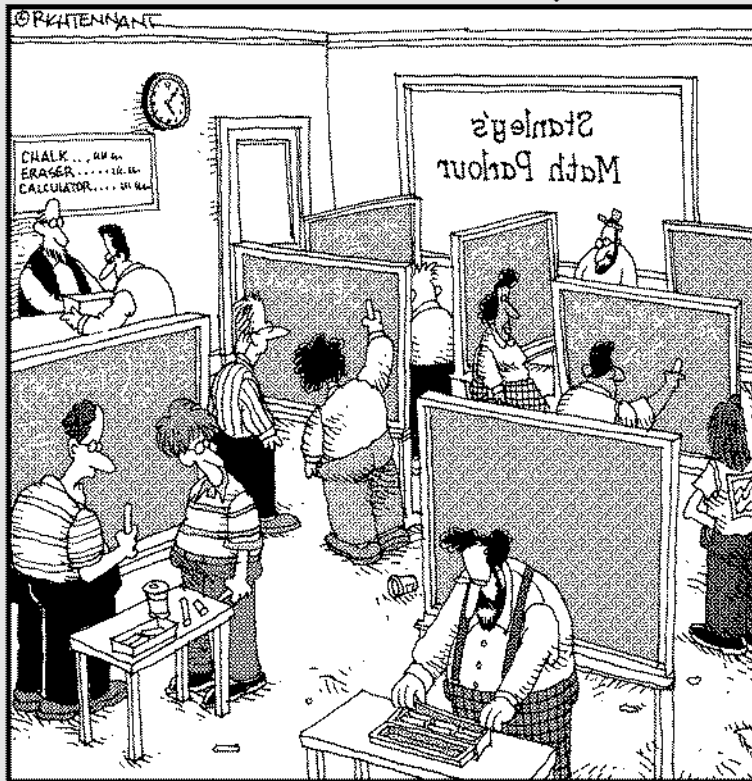


Part II

Indefinite Integrals

The 5th Wave

By Rich Tennant



"It's 'Fast Herschel Fenniman', the most notorious math hustler of all time. If he asks if you'd like to run some trigonometric integrals with him, just walk away."

In this part . . .

You begin calculating the indefinite integral as an anti-derivative — that is, as the inverse of a derivative. In practice, this is easier for some functions than others. So, I show you four important tricks — variable substitution, integration by parts, trig substitution, and integrating with partial fractions — for turning a function you don't know how to integrate into one that you do.

Chapter 4

Instant Integration: Just Add Water (And C)

In This Chapter

- ▶ Calculating simple integrals as anti-derivatives
 - ▶ Using 17 integral formulas and 3 integration rules
 - ▶ Integrating more difficult functions by using more than one integration tool
 - ▶ Clarifying the difference between integrative and nonintegrable functions
-

First the good news: Because integration is the inverse of differentiation, you already know how to evaluate a lot of basic integrals.

Now the bad news: In practice, integration is often a lot trickier than differentiation. I'm telling you this upfront because a) it's true; b) I believe in honesty; and c) you should prepare yourself before your first exam. (Buying and reading this book, by the way, are *great* first steps!)

In this chapter — and also in Chapters 5 through 8 — I focus exclusively on one question: How do you integrate every single function on the planet? Okay, I'm exaggerating, but not by much. I give you a manageable set of integration techniques that you can do with a pencil and paper, and if you know when and how to apply them, you'll be able to integrate everything but the kitchen sink.

First, I show you how to start integrating by thinking about integration as anti-differentiation — that is, as the inverse of differentiation. I give you a not-too-long list of basic integrals, which mirrors the list of basic derivatives from Chapter 2. I also give you a few rules for breaking down functions into manageable chunks that are easier to integrate.

After that, I show you a few techniques for tweaking functions to make them look like the functions you already know how to integrate. By the end of this chapter, you have the tools to integrate dozens of functions quickly and easily.

Evaluating Basic Integrals

In Calculus I (which I cover in Chapter 2), you find that a few algorithms — such as the Product Rule, Quotient Rule, and Chain Rule — give you the tools to differentiate just about every function your professor could possibly throw at you. In Calculus II, students often greet the news that “there’s no Chain Rule for integration” with celebratory cheers. By the middle of the semester, they usually revise this opinion.

Using the 17 basic anti-derivatives for integrating

In Chapter 2, I give you a list of 17 derivatives to know, cherish, and above all *memorize* (yes, I said *memorize*). Reading that list may lead you to believe that I’m one of those harsh über-math dudes who takes pleasure in cruel and unusual curricular activities.

But math is kind of like the Ghost of Christmas Past — the stuff you thought was long ago dead and buried comes back to haunt you. And so it is with derivatives. If you already know them, you’ll find this section easy.

The Fundamental Theorem of Calculus shows that integration is the inverse of differentiation up to a constant C . This key theorem gives you a way to begin integrating. In Table 4-1, I show you how to integrate a variety of common functions by identifying them as the derivatives of functions you already know.

Table 4-1 The 17 Basic Integrals (Anti-Derivatives)	
<i>Derivative</i>	<i>Integral (Anti-Derivative)</i>
$\frac{d}{dx} n = 0$	$\int 0 dx = C$
$\frac{d}{dx} x = 1$	$\int 1 dx = x + C$
$\frac{d}{dx} e^x = e^x$	$\int e^x = e^x + C$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln x + C$
$\frac{d}{dx} n^x = n^x \ln n$	$\int n^x dx = \frac{n^x}{\ln n} + C$
$\frac{d}{dx} \sin x = \cos x$	$\int \cos x dx = \sin x + C$

<i>Derivative</i>	<i>Integral (Anti-Derivative)</i>
$\frac{d}{dx} \cos x = -\sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx} \tan x = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx} \cot x = -\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\frac{d}{dx} \sec x = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx} \csc x = -\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$
$\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$	$\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos x + C$
$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \arctan x + C$
$\frac{d}{dx} \operatorname{arccot} x = -\frac{1}{1+x^2}$	$\int -\frac{1}{1+x^2} \, dx = \operatorname{arccot} x + C$
$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \operatorname{arcsec} x + C$
$\frac{d}{dx} \operatorname{arccsc} x = -\frac{1}{x\sqrt{x^2-1}}$	$\int -\frac{1}{x\sqrt{x^2-1}} \, dx = \operatorname{arccsc} x + C$

As I discuss in Chapter 3, you need to add the constant of integration C because constants differentiate to 0. For example:

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \sin x + 1 = \cos x$$

$$\frac{d}{dx} \sin x - 100 = \cos x$$

So when you integrate by using anti-differentiation, you need to account for the potential presence of this constant:

$$\int \cos x \, dx = \sin x + C$$

Three important integration rules

After you know how to integrate by using the 17 basic anti-derivatives in Table 4-1, you can expand your repertoire with three additional integration

rules: the Sum Rule, the Constant Multiple Rule, and the Power Rule. These three rules mirror those that you know from differentiation.

The Sum Rule for integration

The Sum Rule for integration tells you that integrating long expressions term by term is okay. Here it is formally:

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

For example:

$$\int \left(\cos x + x^2 - \frac{1}{x} \right) dx = \int \cos x dx + \int x^2 dx - \int \frac{1}{x} dx$$

Note that the Sum Rule also applies to expressions of more than two terms. It also applies regardless of whether the term is positive or negative. (Some books call this variation the Difference Rule, but you get the idea.) Splitting this integral into three parts allows you to integrate each separately by using a different anti-differentiation rule:

$$= \sin x + \frac{1}{3} x^3 - \ln x + C$$

Notice that I add only one C at the end. Technically speaking, you should add one variable of integration (say, C_1 , C_2 , and C_3) for each integral that you evaluate. But, at the end, you can still declare the variable $C = C_1 + C_2 + C_3$ to consolidate all these variables. In most cases when you use the Sum Rule, you can skip this step and just tack a C onto the end of the answer.

The Constant Multiple Rule for integration

The Constant Multiple Rule tells you that you can move a constant outside of a derivative before you integrate. Here it is expressed in symbols:

$$\int nf(x) dx = n \int f(x) dx$$

For example:

$$\int 3 \tan x \sec x dx = 3 \int \tan x \sec x dx$$

As you can see, this rule mirrors the Constant Multiple Rule for differentiation. With the constant out of the way, integrating is now easy using an anti-differentiation rule:

$$= 3 \sec x + C$$

The Power Rule for integration

The Power Rule for integration allows you to integrate any real power of x (except -1). Here's the Power Rule expressed formally:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

For example:

$$\int x dx = \frac{1}{2} x^2 + C$$

$$\int x^2 dx = \frac{1}{3} x^3 + C$$

$$\int x^{100} dx = \frac{1}{101} x^{101} + C$$



The Power Rule works fine for negative powers of x , which are powers of x in the denominator. For example:

$$\begin{aligned} \int \frac{1}{x^2} dx \\ &= \int x^{-2} dx \\ &= -x^{-1} + C \\ &= -\frac{1}{x} + C \end{aligned}$$



The Power Rule also works for rational powers of x , which are roots of x . For example:

$$\begin{aligned} \int \sqrt{x^3} dx \\ &= \int x^{\frac{3}{2}} dx \\ &= \frac{2}{5} x^{\frac{5}{2}} + C \\ &= \frac{2}{5} \sqrt{x^5} + C \end{aligned}$$



The *only* real-number power that the Power Rule doesn't work for is -1 . Fortunately, you have an anti-differentiation rule to handle this case:

$$\begin{aligned} \int \frac{1}{x} dx \\ &= \int x^{-1} dx \\ &= \ln |x| + C \end{aligned}$$

What happened to the other rules?

Integration contains formulas that mirror the Sum Rule, the Constant Multiple Rule, and the Power Rule for differentiation. But it lacks formulas that look like the Product Rule, Quotient Rule, and Chain Rule. This fact may sound like good news, but the lack of formulas makes integration a lot trickier in practice than differentiation is.

In fact, Chapters 5 through 8 focus on a bunch of methods that mathematicians have devised for getting around this difficulty. Chapter 5 focuses on variable substitution, which is a limited form of the Chain Rule. And in Chapter 6, I show you integration by parts, which is an adaptation of the Product Rule.

Evaluating More Difficult Integrals

The anti-differentiation rules for integrating, which I explain earlier in this chapter, greatly limit how many integrals you can compute easily. In many cases, however, you can tweak a function to make it easier to integrate.

In this section, I show you how to integrate certain fractions and roots by using the Power Rule. I also show you how to use the trig identities in Chapter 2 to stretch your capacity to integrate trig functions.

Integrating polynomials

You can integrate *any* polynomial in three steps by using the rules from this section:

- 1. Use the Sum Rule to break the polynomial into its terms and integrate each of these separately.**
- 2. Use the Constant Multiple Rule to move the coefficient of each term outside its respective integral.**
- 3. Use the Power Rule to evaluate each integral. (You only need to add a single C to the end of the resulting expression.)**

For example, suppose that you want to evaluate the following integral:

$$\int (10x^6 - 3x^3 + 2x - 5) dx$$

1. Break the expression into four separate integrals:

$$= \int 10x^6 dx - \int 3x^3 dx + \int 2x dx - \int 5 dx$$

2. Move each of the four coefficients outside its respective integral:

$$= 10 \int x^6 dx - 3 \int x^3 dx + 2 \int x dx - 5 \int dx$$

3. Integrate each term separately using the Power Rule:

$$= \frac{10}{7}x^7 - \frac{3}{4}x^4 + x^2 - 5x + C$$



You can integrate *any* polynomial by using this method. Many integration methods I introduce later in this book rely on this fact. So, practice integrating polynomials until you feel so comfortable that you could do it in your sleep.

Integrating rational expressions

In many cases, you can untangle hairy rational expressions and integrate them by using the anti-differentiation rules plus the other three rules in this chapter.

For example, here's an integral that looks like it may be difficult:

$$\int \frac{(x^2 + 5)(x - 3)^2}{\sqrt{x}} dx$$

You can split the function into several fractions, but without the Product Rule or Quotient Rule, you're then stuck. Instead, expand the numerator and put the denominator in exponential form:

$$= \int \frac{x^4 - 6x^3 + 14x^2 - 30x + 45}{x^{\frac{1}{2}}} dx$$

Next, split the expression into five terms:

$$= \int \left(x^{\frac{7}{2}} - 6x^{\frac{5}{2}} + 14x^{\frac{3}{2}} - 30x^{\frac{1}{2}} + 45x^{-\frac{1}{2}} \right) dx$$

Then, use the Sum Rule to separate the integral into five separate integrals and the Constant Multiple Rule to move the coefficient outside the integral in each case:

$$= \int x^{\frac{7}{2}} dx - 6 \int x^{\frac{5}{2}} dx + 14 \int x^{\frac{3}{2}} dx - 30 \int x^{\frac{1}{2}} dx + 45 \int x^{-\frac{1}{2}} dx$$

Now, you can integrate each term separately using the Power Rule:

$$= \frac{2}{9} x^{\frac{9}{2}} - \frac{12}{7} x^{\frac{7}{2}} + \frac{28}{5} x^{\frac{5}{2}} - 20x^{\frac{3}{2}} + 90x^{\frac{1}{2}} + C$$

Using identities to integrate trig functions

At first glance, some products or quotients of trig functions may seem impossible to integrate by using the formulas I give you earlier in this chapter. But, you'll be surprised how much headway you can often make when you integrate an unfamiliar trig function by first tweaking it using the Basic Five trig identities that I list in Chapter 2.

The unseen power of these identities lies in the fact that they allow you to express *any* combination of trig functions into a combination of sines and cosines. Generally speaking, the trick is to simplify an unfamiliar trig function and turn it into something that you know how to integrate.



When you're faced with an unfamiliar product or quotient of trig functions, follow these steps:

1. Use trig identities to turn all factors into sines and cosines.
2. Cancel factors wherever possible.
3. If necessary, use trig identities to eliminate all fractions.

For example:

$$\int \sin^2 x \cot x \sec x \, dx$$

In its current form, you can't integrate this expression by using the rules from this chapter. So you follow these steps to turn it into an expression you can integrate:

1. Use the identities $\cot x = \frac{\cos x}{\sin x}$ and $\sec x = \frac{1}{\cos x}$:

$$= \int \sin^2 x \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} \, dx$$

2. Cancel both $\sin x$ and $\cos x$ in the numerator and denominator:

$$= \int \sin x \, dx$$

In this example, even without Step 3, you have a function that you can integrate.

$$= -\cos x + C$$

Here's another example:

$$\int \tan x \sec x \csc x \, dx$$

Again, this integral looks like a dead end before you apply the five basic trig identities to it:

1. Turn all three factors into sines and cosines:

$$= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\sin x} dx$$

2. Cancel $\sin x$ in the numerator and denominator:

$$= \int \frac{1}{\cos^2 x} dx$$

3. Use the identity $\cos x = \frac{1}{\sec x}$ to eliminate the fraction:

$$= \int \sec^2 x dx$$

$$= \tan x + C$$

Again, you turn an unfamiliar function into one of the ten trig functions that you know how to integrate.

I show you lots more tricks for integrating trig functions in Chapter 7.

Understanding Integrability

By now, you've probably figured out that, in practice, integration is usually harder than differentiation. The lack of any set rules for integrating products, quotients, and compositions of functions makes integration something of an art rather than a science.

So, you may think that a large number of functions are differentiable, with a smaller subset of these being integrable. It turns out that this conclusion is false. In fact, the set of integrable functions is larger, with a smaller subset of these being differentiable. To understand this fact, you need to be clear on what the words *integrable* and *differentiable* really mean.

In this section, I shine some light on two common mistakes that students make when trying to understand what integrability is all about. After that, I discuss what it means for a function to be integrable, and I show you why many functions that are integrable aren't differentiable.

Understanding two red herrings of integrability

In trying to understand what makes a function integrable, you first need to understand two related issues: difficulties in *computing integrals* and *representing integrals as functions*. These issues are valid, but they're red herrings — that is, they don't really affect whether a function is integrable.

Computing integrals

For many input functions, integrals are more difficult to compute than derivatives are. For example, suppose that you want to differentiate and integrate the following function:

$$y = 3x^5 e^{2x}$$

You can differentiate this function easily by using the Product Rule (I take an additional step to simplify the answer):

$$\begin{aligned}\frac{dy}{dx} &= 3 \left[\frac{d}{dx}(x^5) e^{2x} + \frac{d}{dx}(e^{2x}) x^5 \right] \\ &= 3(5x^4 e^{2x} + 2e^{2x} x^5) \\ &= 3x^4 e^{2x}(2x + 5)\end{aligned}$$

Because no such rule exists for integration, in this example you're forced to seek another method. (You find this method in Chapter 6, where I discuss integration by parts.)

Finding solutions to integrals can be tricky business. In comparison, finding derivatives is comparatively simple — you learned most of what you need to know about it in Calculus I.

Representing integrals as functions

Beyond difficulties in computation, the integrals of certain functions simply can't be represented by using the functions that you're used to.

More precisely, some integrals can't be represented as *elementary functions* — that is, as combinations of the functions you know from Pre-Calculus. (See Chapter 14 for a more in-depth look at elementary functions.)

For example, take the following function:

$$y = e^{-x^2}$$

You can find the derivative of the function easily by using the Chain Rule:

$$\begin{aligned}\frac{d}{dx} e^{x^2} &= e^{x^2} \left(\frac{d}{dx} x^2 \right) \\ &= e^{x^2} (2x) \\ &= 2xe^{x^2}\end{aligned}$$

However, the integral of the same function, e^{x^2} , can't be expressed as a function — at least, not any function that you're used to.

Instead, you can express this integral either *exactly* — as an infinite series — or *approximately* — as a function that approximates the integral to a given level of precision. (See Part IV for more on infinite series.) Alternatively, you can just leave it as an integral, which also expresses it just fine for some purposes.

Understanding what integrable really means

When mathematicians discuss whether a function is integrable, they aren't talking about the difficulty of computing that integral — or even whether a method has been discovered. Each year, mathematicians find new ways to integrate classes of functions. However, this fact doesn't mean that previously nonintegrable functions are now integrable.

Similarly, a function's integrability also doesn't hinge upon whether its integral can be easily represented as another function, without resorting to infinite series.

In fact, when mathematicians say that a function is integrable, they mean only that the integral is *well defined* — that is, that it makes mathematical sense.



In practical terms, integrability hinges on continuity: If a function is continuous on a given interval, it's integrable on that interval. Additionally, if a function has only a finite number of discontinuities on an interval, it's also integrable on that interval.

You probably remember from Calculus I that many functions — such as those with discontinuities, sharp turns, and vertical slopes — are nondifferentiable. Discontinuous functions are also nonintegrable. However, functions with sharp turns and vertical slopes are integrable.

For example, the function $y = |x|$ contains a sharp point at $x = 0$, so the function is nondifferentiable at this point. However, the same function is integrable for all values of x . This is just one of infinitely many examples of a function that's integrable but not differentiable in the entire set of real numbers.

So, surprisingly, the set of differentiable functions is actually a subset of the set of integrable functions. In practice, however, computing the integral of most functions is more difficult than computing the derivative.

Chapter 5

Making a Fast Switch: Variable Substitution

In This Chapter

- ▶ Understanding how variable substitution works
 - ▶ Recognizing when variable substitution can help you
 - ▶ Knowing a shortcut for using substitution with definite integrals
-

Unlike differentiation, integration doesn't have a Chain Rule. This fact makes integrating *compositions of functions* (functions within functions) a little bit tricky. The most useful trick for integrating certain common compositions of functions uses variable substitution.

With *variable substitution*, you set a variable (usually u) equal to part of the function that you're trying to integrate. The result is a simplified function that you can integrate by using the anti-differentiation formulas and the three basic integration rules (Sum Rule, Constant Multiple Rule, and Power Rule — all discussed in Chapter 4).

In this chapter, I show you how to use variable substitution. Then I show you how to identify a few common situations where variable substitution is helpful. After you get comfortable with the process, I give you a quick way to integrate by just looking at the problem and writing down the answer. Finally, I show you how to skip a step when using variable substitution to evaluate definite integrals.

Knowing How to Use Variable Substitution

The anti-differentiation formulas plus the Sum Rule, Constant Multiple Rule, and Power Rule (all discussed in Chapter 4) allow you to integrate a variety of common functions. But as functions begin to get a little bit more complex, these methods become insufficient. For example, these methods don't work on the following:

$$\int \sin 2x \, dx$$

To evaluate this integral, you need some stronger medicine. The sticking point here is the presence of the constant 2 inside the sine function. You have an anti-differentiation rule for integrating the sine of a variable, but how do you integrate the sine of a variable times a constant?

The answer is variable substitution, a five-step process that allows you to integrate where no integral has gone before:

- 1. Declare a variable u and set it equal to an algebraic expression that appears in the integral, and then substitute u for this expression in the integral.**
- 2. Differentiate u to find $\frac{du}{dx}$, and then isolate all x variables on one side of the equal sign.**
- 3. Make another substitution to change dx and all other occurrences of x in the integral to an expression that includes du .**
- 4. Integrate by using u as your new variable of integration.**
- 5. Express this answer in terms of x .**

I don't expect these steps to make much sense until you see how they work in action. In the rest of this section, I show you how to use variable substitution to solve problems that you wouldn't be able to integrate otherwise.

Finding the integral of nested functions

Suppose that you want to integrate the following:

$$\int \sin 2x \, dx$$

The difficulty here lies in the fact that this function is the composition of two functions: the function $2x$ nested inside a sine function. If you were differentiating, you could use the Chain Rule. Unfortunately, no Chain Rule exists for integration.

Fortunately, this function is a good candidate for variable substitution. Follow the five steps I give you in the previous section:

1. Declare a new variable u as follows and substitute it into the integral:

$$\text{Let } u = 2x$$

Now, substitute u for $2x$ as follows:

$$\int \sin 2x \, dx = \int \sin u \, dx$$

This may look like the answer to all your troubles, but you have one more problem to resolve. As it stands, the symbol dx tells you that variable of integration is still x .

To integrate properly, you need to find a way to change dx to an expression containing du . That's what Steps 2 and 3 are about.

2. Differentiate the function $u = 2x$ and isolate the x terms on one side of the equal sign:

$$\frac{du}{dx} = 2$$

Now, treat the symbol $\frac{du}{dx}$ as if it's a fraction, and isolate the x terms on one side of the equal sign. I do this in two steps:

$$du = 2 \, dx$$

$$\frac{1}{2} \, du = dx$$

3. Substitute $\frac{1}{2}du$ for dx into the integral:

$$\int \sin u \left(\frac{1}{2} \, du \right)$$

You can treat the $\frac{1}{2}$ just like any coefficient and use the Constant Multiple Rule to bring it outside the integral:

$$= \frac{1}{2} \int \sin u \, du$$

4. At this point, you have an expression that you know how to evaluate:

$$= -\frac{1}{2} \cos u + C$$

5. Now that the integration is done, the last step is to substitute $2x$ back in for u :

$$-\frac{1}{2} \cos 2x + C$$

You can check this solution by differentiating using the Chain Rule:

$$\begin{aligned} & \frac{d}{dx} \left(-\frac{1}{2} \cos 2x + C \right) \\ &= \frac{d}{dx} \left(-\frac{1}{2} \cos 2x \right) + \frac{d}{dx} C \\ &= -\frac{1}{2} (-\sin 2x) (2) + 0 \\ &= \sin 2x \end{aligned}$$

Finding the integral of a product

Imagine that you're faced with this integral:

$$\int \sin^3 x \cos x \, dx$$

The problem in this case is that the function that you're trying to integrate is the product of two functions — $\sin^3 x$ and $\cos x$. This would be simple to differentiate with the Product Rule, but integration doesn't have a Product Rule. Again, variable substitution comes to the rescue:

1. **Declare a variable as follows and substitute it into the integral:**

$$\text{Let } u = \sin x$$

You may ask *how* I know to declare u equal to $\sin x$ (rather than, say, $\sin^3 x$ or $\cos x$). I answer this question later in the chapter. For now, just follow along and get the mechanics of variable substitution.

You can substitute this variable into the expression that you want to integrate as follows:

$$\int \sin^3 x \cos x \, dx = \int u^3 \cos x \, dx$$

Notice that the expression $\cos x \, dx$ still remains and needs to be expressed in terms of u .

2. **Differentiate the function $u = \sin x$ and isolate the x variables on one side of the equal sign:**

$$\frac{du}{dx} = \cos x$$

Isolate the x variables on one side of the equal sign:

$$du = \cos x \, dx$$

3. Substitute du for $\cos x \, dx$ in the integral:

$$\int u^3 \, du$$

4. Now you have an expression that you can integrate:

$$= \frac{1}{4} u^4 + C$$

5. Substitute $\sin x$ for u :

$$= \frac{1}{4} \sin^4 x + C$$

And again, you can check this answer by differentiating with the Chain Rule:

$$\begin{aligned} & \frac{d}{dx} \left(\frac{1}{4} \sin^4 x + C \right) \\ &= \frac{d}{dx} \frac{1}{4} \sin^4 x + \frac{d}{dx} C \\ &= \frac{1}{4} (4 \sin^3 x) (\cos x) + 0 \\ &= \sin^3 x \cos x \end{aligned}$$

This derivative matches the original function, so the integration is correct.

Integrating a function multiplied by a set of nested functions

Suppose that you want to integrate the following:

$$\int x \sqrt{3x^2 + 7} \, dx$$

This time, you're trying to integrate the product of a function (x) and a composition of functions (the function $3x^2 + 7$ nested inside a square root function). If you were differentiating, you could use a combination of the Product Rule and the Chain Rule, but these options aren't available for integration. Here's how you integrate, step by step, by using variable substitution:

1. Declare a variable u as follows and substitute it into the integral:

$$\text{Let } u = 3x^2 + 7$$

Here, you may ask how I know what value to assign to u . Here's the short answer: u is the inner function, as you would identify if you were using the Chain Rule. (See Chapter 2 for a review of the Chain Rule.) I explain this more fully later in "Recognizing When to Use Substitution."

Now, substitute u into the integral:

$$\int x \sqrt{3x^2 + 7} \, dx = \int x \sqrt{u} \, dx$$

Make one more small rearrangement to place all the remaining x terms together:

$$= \int \sqrt{u} x dx$$

This rearrangement makes clear that I still have to find a substitution for $x dx$.

2. Now differentiate the function $u = 3x^2 + 7$:

$$\frac{du}{dx} = 6x$$

From Step 1, I know that I need to replace $x dx$ in the integral:

$$du = 6x dx$$

$$\frac{1}{6} du = x dx$$

3. Substitute $\frac{du}{6}$ for $x dx$:

$$= \int \sqrt{u} \left(\frac{1}{6} du \right)$$

You can move the fraction $\frac{1}{6}$ outside the integral:

$$= \frac{1}{6} \int \sqrt{u} du$$

4. Now you have an integral that you know how to evaluate.

I take an extra step, putting the square root in exponential form, to make sure that you see how to do this:

$$= \frac{1}{6} \int u^{\frac{1}{2}} du$$

$$= \frac{1}{6} \left(\frac{2}{3} \right) u^{\frac{3}{2}} + C$$

$$= \frac{1}{9} u^{\frac{3}{2}} + C$$

5. To finish up, substitute $3x^2 + 7$ for u :

$$= \frac{1}{9} (3x^2 + 7)^{\frac{3}{2}} + C$$

As with the first two examples in this chapter, you can always check your integration by differentiating the result:

$$\begin{aligned} & \frac{d}{dx} \left[\frac{1}{9} (3x^2 + 7)^{\frac{3}{2}} + C \right] \\ &= \frac{d}{dx} \frac{1}{9} (3x^2 + 7)^{\frac{3}{2}} + \frac{d}{dx} C \\ &= \frac{1}{9} \left(\frac{3}{2} \right) (3x^2 + 7)^{\frac{1}{2}} (6x) + 0 \\ &= x \sqrt{3x^2 + 7} \end{aligned}$$

As if by magic, the derivative brings you back to the function you started with.

Recognizing When to Use Substitution

In the previous section, I show you the mechanics of variable substitution — that is, *how* to perform variable substitution. In this section, I clarify *when* to use variable substitution.

You may be able to use variable substitution in three common situations. In these situations, the expression you want to evaluate is one of the following:

- ✓ A composition of functions — that is, a function nested in a function
- ✓ A function multiplied by a function
- ✓ A function multiplied by a computation of functions

Integrating nested functions

Compositions of functions — that is, one function nested inside another — are of the form $f(g(x))$. You can integrate them by substituting $u = g(x)$ when

- ✓ You know how to integrate the outer function f .
- ✓ The inner function $g(x)$ differentiates to a constant — that is, it's of the form ax or $ax + b$.

Example #1

Here's an example. Suppose that you want to integrate the function

$$\csc^2(4x + 1) dx$$

Again, this is a composition of two functions:

- ✓ The outer function f is the \csc^2 function, which you know how to integrate.
- ✓ The inner function is $g(x) = 4x + 1$, which differentiates to the constant 4.

This time the composition is held together by the equality $u = 4x + 1$. That is, the two basic functions $f(u) = \csc^2 u$ and $g(x) = 4x + 1$ are composed by the equality $u = 4x + 1$ to produce the function $f(g(x)) = \csc^2(4x + 1)$.

Both criteria are met, so this integral is another prime candidate for substitution using $u = 4x + 1$. Here's how you do it:

1. Declare a variable u and substitute it into the integral:

$$\text{Let } u = 4x + 1$$

$$\int \csc^2(4x + 1) dx = \int \csc^2 u dx$$



2. Differentiate $u = 4x + 1$ and isolate the x term:

$$\frac{du}{dx} = 4$$

$$\frac{du}{4} = dx$$

3. Substitute $\frac{du}{4}$ for dx in the integral:

$$\int \csc^2 u \left(\frac{1}{4} du \right)$$

$$= \frac{1}{4} \int \csc^2 u du$$

4. Evaluate the integral:

$$= -\frac{1}{4} \cot u + C$$

5. Substitute back $4x + 1$ for u :

$$= -\frac{1}{4} \cot(4x + 1) + C$$

Example #2

Here's one more example. Suppose that you want to evaluate the following integral:

$$\int \frac{1}{x-3} dx$$

Again, this is a composition of two functions:

- ✓ The outer function f is a fraction — technically, an exponent of -1 — which you know how to integrate.
- ✓ The inner function is $g(x) = x - 3$, which differentiates to 1.



Here, the composition is held together by the equality $u = x - 3$. That is, the two basic functions $f(u) = \frac{1}{u}$ and $g(x) = x - 3$ are composed by the equality $u = x - 3$ to produce the function $f(g(x)) = \frac{1}{x-3}$.

The criteria are met, so you can integrate by using the equality $u = x - 3$:

1. Declare a variable u and substitute it into the integral:

$$\text{Let } u = x - 3$$

$$\int \frac{1}{x-3} dx = \int \frac{1}{u} dx$$

2. Differentiate $u = x - 3$ and isolate the x term:

$$\frac{du}{dx} = 1$$

$$du = dx$$

3. Substitute du for dx in the integral:

$$\int \frac{1}{u} du$$

4. Evaluate the integral:

$$= \ln |u| + C$$

5. Substitute back $x - 3$ for u :

$$= \ln |x - 3| + C$$

Knowing a shortcut for nested functions

After you work through enough examples of variable substitution, you may begin to notice certain patterns emerging. As you get more comfortable with the concept, you can use a shortcut to integrate compositions of functions — that is, nested functions of the form $f(g(x))$. Technically, you're using the variable substitution $u = g(x)$, but you can bypass this step and still get the right answer.

This shortcut works for compositions of functions $f(g(x))$ for which

- ✔ You know how to integrate the outer function f .
- ✔ The inner function $g(x)$ is of the form ax or $ax + b$ — that is, it differentiates to a constant.

When these two conditions hold, you can integrate $f(g(x))$ by using the following three steps:

1. Write down the reciprocal of the coefficient of x .
2. Multiply by the integral of the outer function, copying the inner function as you would when using the Chain Rule in differentiation.
3. Add C .

Example #1

For example:

$$\int \cos 4x dx$$

Notice that this is a function nested within a function, where the following are true:

- ✔ The outer function f is the cosine function, which you know how to integrate.
- ✔ The inner function is $g(x) = 4x$, which is of the form ax .

So, you can integrate this function quickly as follows:

1. Write down the reciprocal of 4 — that is, $\frac{1}{4}$:

$$\frac{1}{4}$$

2. Multiply this reciprocal by the integral of the outer function, copying the inner function:

$$\frac{1}{4} \sin 4x$$

3. Add C:

$$\frac{1}{4} \sin 4x + C$$

That's it! You can check this easily by differentiating, using the Chain Rule:

$$\begin{aligned} & \frac{d}{dx} \left(\frac{1}{4} \sin 4x + C \right) \\ &= \frac{1}{4} \cos 4x (4) \\ &= \cos 4x \end{aligned}$$

Example #2

Here's another example:

$$\int \sec^2 10x \, dx$$

Remember as you begin that $\sec^2 10x \, dx$ is a notational shorthand for $[\sec(10x)]^2$. So, the outer function f is the \sec^2 function and the inner function is $g(x) = 10x$. (See Chapter 2 for more on the ins and outs of trig notation.) Again, the criteria for variable substitution are met:

1. Write down the reciprocal of 10 — that is, $\frac{1}{10}$:

$$\frac{1}{10}$$

2. Multiply this reciprocal by the integral of the outer function, copying the inner function:

$$\frac{1}{10} \tan 10x$$

3. Add C:

$$\frac{1}{10} \tan 10x + C$$

Here's the check:

$$\begin{aligned} & \frac{d}{dx} \left(\frac{1}{10} \tan 10x + C \right) \\ &= \frac{d}{dx} \frac{1}{10} \tan 10x + \frac{d}{dx} C \\ &= \frac{1}{10} \sec^2 10x (10) + 0 \\ &= \sec^2 10x \end{aligned}$$

Example #3

Here's another example:

$$\int \frac{1}{7x+2} dx$$

In this case, the outer function is division, which counts as a function, as I explain earlier in “Recognizing When to Use Substitution.” The inner function is $7x + 2$. Both of these functions meet the criteria, so here's how to perform this integration:

1. Write down the reciprocal of the coefficient 7 — that is, $\frac{1}{7}$:

$$\frac{1}{7}$$

2. Multiply this reciprocal by the integral of the outer function, copying the inner function:

$$\frac{1}{7} \ln |7x + 2|$$

3. Add C :

$$\frac{1}{7} \ln |7x + 2| + C$$

You're done! As always, you can check your result by differentiating, using the Chain Rule:

$$\begin{aligned} & \frac{d}{dx} \left(\frac{1}{7} \ln |7x + 2| + C \right) \\ &= \frac{1}{7} \left(\frac{1}{7x + 2} \right) (7) \\ &= \frac{1}{7x + 2} \end{aligned}$$

Example #4

Here's one more example:

$$\int \sqrt{12x - 5} \, dx$$

This time, the outer function f is a square root — that is, an exponent of $\frac{1}{2}$ — and $g(x) = 12x - 5$, so you can use a quick substitution:

1. Write down the reciprocal of 12 — that is, $\frac{1}{12}$:

$$\frac{1}{12}$$

2. Multiply the integral of the outer function, copying down the inner function:

$$\begin{aligned} & \frac{1}{12} \frac{2}{3} (12x - 5)^{\frac{3}{2}} \\ &= \frac{1}{18} (12x - 5)^{\frac{3}{2}} \end{aligned}$$

3. Add C :

$$\frac{1}{18} (12x - 5)^{\frac{3}{2}} + C$$

Table 5-1 gives you a variety of integrals in this form. As you look over this chart, get a sense of the pattern so that you can spot it when you have an opportunity to integrate quickly.

Integral	Evaluation
$\int e^{5x} \, dx$	$\frac{1}{5} e^{5x} + C$
$\int \sin 7x \, dx$	$-\frac{1}{7} \cos 7x + C$
$\int \sec^2 \frac{X}{3} \, dx$	$3 \tan \frac{X}{3} + C$
$\int \tan 8x \sec 8x \, dx$	$\frac{1}{8} \sec 8x + C$
$\int e^{5x+2} \, dx$	$\frac{1}{5} e^{5x+2} + C$
$\int \cos(x-4) \, dx$	$\sin(x-4) + C$

Substitution when one part of a function differentiates to the other part

When $g'(x) = f(x)$, you can use the substitution $u = g(x)$ to integrate the following:

- ✓ Expressions of the form $f(x) \cdot g(x)$
- ✓ Expressions of the form $f(x) \cdot h(g(x))$, provided that h is a function that you already know how to integrate

Don't worry if you don't understand all this math-ese. In the following sections, I show you how to recognize both of these cases and integrate each. As usual, variable substitution helps to fill the gaps left by the absence of a Product Rule and a Chain Rule for integration.

Expressions of the form $f(x) \cdot g(x)$

Some products of functions yield quite well to variable substitution. Look for expressions of the form $f(x) \cdot g(x)$ where

- ✓ You know how to integrate $g(x)$.
- ✓ The function $f(x)$ is the derivative of $g(x)$.

For example:

$$\int \tan x \sec^2 x \, dx$$

The main thing to notice here is that the derivative of $\tan x$ is $\sec^2 x$. This is a great opportunity to use variable substitution:

1. Declare u and substitute it into the integral:

$$\text{Let } u = \tan x$$

$$\int \tan x \sec^2 x \, dx = \int u \sec^2 x \, dx$$

2. Differentiate as planned:

$$\frac{du}{dx} = \sec^2 x$$

$$du = \sec^2 x \, dx$$

3. Perform another substitution:

$$= \int u \, du$$

4. This integration couldn't be much easier:

$$= \frac{1}{2} u^2 + C$$

5. Substitute back $\tan x$ for u :

$$= \frac{1}{2} \tan^2 x + C$$

Expressions of the form $f(x) \cdot h(g(x))$

Here's a hairy-looking integral that actually responds well to substitution:

$$\int \frac{(2x+1)}{(x^2+x-5)^{\frac{4}{3}}} dx$$

The key insight here is that the numerator of this fraction is the derivative of the inner function in the denominator. Watch how this plays out in this substitution:

1. Declare u equal to the denominator and make the substitution:

$$\text{Let } u = x^2 + x - 5$$

Here's the substitution:

$$= \int \frac{2x+1}{u^{\frac{4}{3}}} dx$$

2. Differentiate u :

$$\frac{du}{dx} = 2x + 1$$

$$du = (2x + 1) dx$$

3. The second part of the substitution now becomes clear:

$$= \int \frac{1}{u^{\frac{4}{3}}} du$$

Notice how this substitution hinges on the fact that the numerator is the derivative of the denominator. (You may think that this is quite a coincidence, but coincidences like these happen all the time on exams!)

4. Integration is now quite straightforward:

I take an extra step to remove the fraction before I integrate.

$$= \int u^{-\frac{4}{3}} du$$

$$= -3u^{-\frac{1}{3}} + C$$

5. Substitute back $x^2 + x - 5$ for u :

$$= -3(x^2 + x - 5)^{-\frac{1}{3}} + C$$

Checking the answer by differentiating with the Chain Rule reveals how this problem was set up in the first place:

$$\begin{aligned} & \frac{d}{dx} [-3(x^2 + x - 5)^{-\frac{1}{3}} + C] \\ &= (x^2 + x - 5)^{-\frac{4}{3}} (2x + 1) \\ &= \frac{2x + 1}{(x^2 + x - 5)^{\frac{4}{3}}} \end{aligned}$$

By now, if you've worked through the examples in this chapter, you're probably seeing opportunities to make variable substitutions. For example:

$$\int x^3 \sqrt{x^4 - 1} \, dx$$

Notice that the derivative of $x^4 - 1$ is x^3 , off by a constant factor. So here's the declaration, followed by the differentiation:

$$\begin{aligned} \text{Let } u &= x^4 - 1 \\ \frac{du}{dx} &= 4x^3 \\ \frac{du}{4} &= x^3 \, dx \end{aligned}$$

Now you can just do both substitutions at once:

$$\begin{aligned} & \int \sqrt{u} \cdot \left(\frac{1}{4} du\right) \\ &= \frac{1}{4} \int \sqrt{u} \, du \end{aligned}$$

At this point, you can solve the integral simply — I'll leave this as an exercise for you!

Similarly, here's another example:

$$\int \csc^2 x e^{\cot x} \, dx$$

At first glance, this integral looks just plain horrible. But on further inspection, notice that the derivative of $\cot x$ is $-\csc^2 x$, so this looks like another good candidate:

$$\begin{aligned} \text{Let } u &= \cot x \\ \frac{du}{dx} &= -\csc^2 x \\ -du &= \csc^2 x \, dx \end{aligned}$$

This results in the following substitution:

$$\begin{aligned} &= \int e^u(-du) \\ &= -\int e^u du \end{aligned}$$

Again, this is another integral that you can solve.

Using Substitution to Evaluate Definite Integrals

In the first two sections of this chapter, I cover how and when to evaluate indefinite integrals with variable substitution. All this information also applies to evaluating definite integrals, but I also have a timesaving trick that you should know.

When using variable substitution to evaluate a definite integral, you can save yourself some trouble at the end of the problem. Specifically, you can leave the solution in terms of u by changing the limits of integration.

For example, suppose that you're evaluating the following definite integral:

$$\int_{x=0}^{x=1} x\sqrt{x^2+1} dx$$

Notice that I give the limits of integration as $x = 0$ and $x = 1$. This is just a notational change to remind you that the limits of integration are values of x . This fact becomes important later in the problem.

You can evaluate this equation simply by using variable substitution.

If you're not sure why this substitution works, read the section "Recognizing When to Use Substitution" earlier in this chapter. Follow Steps 1 through 3 of variable substitution:

$$\begin{aligned} \text{Let } u &= x^2 + 1 \\ \frac{du}{dx} &= 2x \\ \frac{du}{2} &= x dx \\ &= \frac{1}{2} \int_{x=0}^{x=2} \sqrt{u} du \end{aligned}$$

If this were an indefinite integral, you'd be ready to integrate. But because this is a definite integral, you still need to express the limits of integration in terms of u rather than x . Do this by substituting values 0 and 1 for x in the substitution equation $u = x^2 + 1$:

$$u = 1^2 + 1 = 2$$

$$u = 0^2 + 1 = 1$$

Now use these values of u as your new limits of integration:

$$= \frac{1}{2} \int_{v=1}^{v=2} \sqrt{u} \, du$$

At this point, you're ready to integrate:

$$= \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{u=1}^{u=2}$$

$$= \frac{1}{3} u^{\frac{3}{2}} \Big|_{v=1}^{v=2}$$

Because you changed the limits of integration, you can now find the answer without switching the variable back to x :

$$= \frac{1}{3} \left(2^{\frac{3}{2}} - 1^{\frac{3}{2}} \right)$$

$$= \frac{1}{3} (\sqrt{8} - 1)$$

$$= \frac{\sqrt{8}}{3} - \frac{1}{3}$$

Chapter 6

Integration by Parts

In This Chapter

- ▶ Making the connection between the Product Rule and integration by parts
 - ▶ Knowing how and when integration by parts works
 - ▶ Integrating by parts by using the DI-agonal method
 - ▶ Practicing the DI-agonal method on the four most common products of functions
-

In Calculus I, you find that the Product Rule allows you to find the derivative of any two functions that are multiplied together. (I review this in Chapter 2, in case you need a refresher.) But integrating the product of two functions isn't quite as simple. Unfortunately, no formula allows you to integrate the product of any two functions. As a result, a variety of techniques have been developed to handle products of functions on a case-by-case basis.

In this chapter, I show you the most widely applicable technique for integrating products, called *integration by parts*. First, I demonstrate how the formula for integration by parts follows the Product Rule. Then I show you how the formula works in practice. After that, I give you a list of the products of functions that are likely to yield to this method.

After you understand the principle behind integration by parts, I give you a method — called the *DI-agonal method* — for performing this calculation efficiently and without errors. Then I show you examples of how to use this method to integrate the four most common products of functions.

Introducing Integration by Parts

Integration by parts is a happy consequence of the Product Rule (discussed in Chapter 2). In this section, I show you how to tweak the Product Rule to derive the formula for integration by parts. I show you two versions of this formula — a complicated version and a simpler one — and then recommend that you memorize the second. I show you how to use this formula, and then I give you a heads up as to when integration by parts is likely to work best.

Reversing the Product Rule

The Product Rule (see Chapter 2) enables you to differentiate the product of two functions:

$$\frac{d}{dx} [f(x) \cdot g(x)] = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

Through a series of mathematical somersaults, you can turn this equation into a formula that's useful for integrating. This derivation doesn't have any truly difficult steps, but the notation along the way is mind-deadening, so don't worry if you have trouble following it. Knowing how to derive the formula for integration by parts is less important than knowing when and how to use it, which I focus on in the rest of this chapter.

The first step is simple: Just rearrange the two products on the right side of the equation:

$$\frac{d}{dx} [f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Next, rearrange the terms of the equation:

$$f(x) \cdot g'(x) = \frac{d}{dx} [f(x) \cdot g(x)] - g(x) \cdot f'(x)$$

Now, integrate both sides of this equation:

$$\int f(x) g'(x) dx = \int \left\{ \frac{d}{dx} [f(x) g(x)] - g(x) f'(x) \right\} dx$$

Use the Sum Rule to split the integral on the left in two:

$$\int f(x) g'(x) dx = \int \frac{d}{dx} [f(x) g(x)] dx - \int g(x) f'(x) dx$$

The first of these two integrals undoes the derivative:

$$\int f(x) g'(x) dx = f(x) g(x) - \int g(x) f'(x) dx$$

This is the formula for integration by parts. But because it's so hairy looking, the following substitution is used to simplify it:

$$\begin{array}{ll} \text{Let } u = f(x) & \text{Let } v = g(x) \\ du = f'(x) dx & dv = g'(x) dx \end{array}$$



Here's the friendlier version of the same formula, which you should memorize:

$$\int u dv = uv - \int v du$$

Knowing how to integrate by parts

The formula for integration by parts gives you the option to break the product of two functions down to its factors and integrate it in an altered form.

To integrate by parts:

1. Decompose the entire integral (including dx) into two factors.
2. Let the factor without dx equal u and the factor with dx equal dv .
3. Differentiate u to find du , and integrate dv to find v .
4. Use the formula $\int u dv = uv - \int v du$.
5. Evaluate the right side of this equation to solve the integral.

For example, suppose that you want to evaluate this integral:

$$\int x \ln x dx$$

In its current form, you can't perform this computation, so integrate by parts:

1. Decompose the integral into $\ln x$ and $x dx$.
2. Let $u = \ln x$ and $dv = x dx$.
3. Differentiate $\ln x$ to find du and integrate $x dx$ to find v :

$$\begin{array}{ll} \text{Let } u = \ln x & \text{Let } dv = x dx \\ \frac{du}{dx} = \frac{1}{x} & \int dv = \int x dx \\ du = \frac{1}{x} dx & v = \frac{1}{2} x^2 \end{array}$$

4. Using these values for u , du , v , and dv , you can use the formula for integration by parts to rewrite the integral as follows:

$$\int x \ln x dx = (\ln x) \left(\frac{1}{2} x^2 \right) - \int \left(\frac{1}{2} x^2 \right) \left(\frac{1}{x} \right) dx$$

At this point, algebra is useful to simplify the right side of the equation:

$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx$$

5. Evaluate the integral on the right:

$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \left(\frac{1}{2} \right) x^2 + C$$

You can simplify this answer just a bit:

$$= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C$$

Therefore, $\int x \ln x \, dx$. To check this answer, differentiate it by using the Product Rule:

$$\begin{aligned} & \frac{d}{dx} \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \right) \\ &= \frac{1}{2} \left[\left(\frac{d}{dx} x^2 \right) \ln x + \left(\frac{d}{dx} \ln x \right) x^2 \right] - \frac{1}{4} 2x \\ &= \frac{1}{2} \left[2x \ln x + \left(\frac{1}{x} \cdot x^2 \right) \right] - \frac{1}{2} x \end{aligned}$$

Now, simplify this result to show that it's equivalent to the function you started with:

$$= x \ln x + \frac{1}{2} x - \frac{1}{2} x = x \ln x$$

Knowing when to integrate by parts

After you know the basic mechanics of integrating by parts, as I show you in the previous section, it's important to recognize when integrating by parts is useful.

To start off, here are two important cases when integration by parts is definitely the way to go:

- ✓ The logarithmic function $\ln x$
- ✓ The first four inverse trig functions ($\arcsin x$, $\arccos x$, $\arctan x$, and $\operatorname{arccot} x$)

Beyond these cases, integration by parts is useful for integrating the product of more than one function. For example:

- ✓ $x \ln x$
- ✓ $x \operatorname{arcsec} x$
- ✓ $x^2 \sin x$
- ✓ $e^x \cos x$

Notice that in each case, you can recognize the product of functions because the variable x appears more than once in the function.



Whenever you're faced with integrating the product of functions, consider variable substitution (which I discuss in Chapter 5) before you think about integration by parts. For example, $x \cos(x^2)$ is a job for variable substitution, not integration by parts. (To see why, flip to Chapter 5.)

When you decide to use integration by parts, your next question is how to split up the function and assign the variables u and dv . Fortunately, a helpful mnemonic exists to make this decision: **L**ovely **I**ntegrals **A**re **T**errific, which stands for **L**ogarithmic, **I**nverse trig, **A**lgebraic, **T**rig. (If you prefer, you can also use the mnemonic **L**ousy **I**ntegrals **A**re **T**errible.) Always choose the *first* function in this list as the factor to set equal to u , and then set the rest of the product (including dx) equal to dv .

You can use integration by parts to integrate any of the functions listed in Table 6-1.

<i>Function</i>	<i>Example</i>	<i>Differentiate u to Find du</i>	<i>Integrate dv to Find v</i>
Log function	$\int \ln x \, dx$	$\ln x$	dx
Log times algebraic	$\int x^4 \ln x \, dx$	$\ln x$	$x^4 \, dx$
Log composed with algebraic	$\int \ln x^3 \, dx$	$\ln x^3$	dx
Inverse trig forms	$\int \arcsin x \, dx$	$\arcsin x$	dx
Algebraic times sine	$\int x^2 \sin x \, dx$	x^2	$\sin x \, dx$
Algebraic times cosine	$\int 3x^5 \cos x \, dx$	$3x^5$	$\sin x \, dx$
Algebraic times exponential	$\int \frac{1}{2} x^2 e^{3x} \, dx$	$\frac{1}{2} x^2$	$e^3 \, dx$
Sine times exponential	$\int e^{\frac{x}{2}} \sin x \, dx$	$e^{\frac{x}{2}}$	$\sin x \, dx$
Cosine times exponential	$\int e^x \cos x \, dx$	e^x	$\cos x \, dx$



When you're integrating by parts, here's the most basic rule when deciding which term to integrate and which to differentiate: If you only know how to integrate one of the two, that's the one you integrate!

Integrating by Parts with the DI-agonal Method

The DI-agonal method is basically integration by parts with a chart that helps you organize information. This method is especially useful when you need to integrate by parts more than once to solve a problem. In this section, I show you how to use the DI-agonal method to evaluate a variety of integrals.

Looking at the DI-agonal chart

The DI-agonal method avoids using u and dv , which are easily confused (especially if you write the letters u and v as sloppily as I do!). Instead, a column for *differentiation* is used in place of u , and a column for *integration* replaces dv .

Use the following chart for the DI-agonal method:

	I
D	
+	
-	

As you can see, the chart contains two columns: the D column for *differentiation*, which has a plus sign and a minus sign, and the I column for *integration*. You may also notice that the D and the I are placed *diagonally* in the chart — yes, the name *DI-agonal method* works on two levels (so to speak).

Using the DI-agonal method

Earlier in this chapter, I provide a list of functions that you can integrate by parts. The DI-agonal method works for all these functions. I also give you the mnemonic **L**ovely **I**ntegrals **A**re **T**errific (which stands for **L**ogarithmic, **I**nverse trig, **A**lgebraic, **T**rig) to help you remember how to assign values of u and dv — that is, what to differentiate and what to integrate.

To use the DI-agonal method:

1. Write the value to differentiate in the box below the D and the value to integrate (omitting the dx) in the box below the I .
2. Differentiate down the D column and integrate down the I column.
3. Add the products of all *full* rows as terms.

I explain this step in further detail in the examples that follow.

4. Add the integral of the product of the two lowest diagonally adjacent boxes.

I also explain this step in greater detail in the examples.

Don't spend too much time trying to figure this out. The upcoming examples show you how it's done and give you plenty of practice. I show you how to use the DI-agonal method to integrate products that include logarithmic, inverse trig, algebraic, and trig functions.

L is for logarithm

You can use the DI-agonal method to evaluate the product of a log function and an algebraic function. For example, suppose that you want to evaluate the following integral:

$$\int x^2 \ln x \, dx$$



Whenever you integrate a product that includes a log function, the log function always goes in the D column.

1. Write the log function in the box below the D and the rest of the function value (omitting the dx) in the box below the I .

	I
D	x^2
+ $\ln x$	
-	

2. Differentiate $\ln x$ and place the answer in the D column.

Notice that in this step, the minus sign already in the box attaches to $\frac{1}{x}$.

	I
D	x^2
+ $\ln x$	
- $\frac{1}{x}$	

3. Integrate x^2 and place the answer in the *I* column.

	I
D	x^2
+ $\ln x$	$\frac{1}{3} x^3$
- $\frac{1}{x}$	

4. Add the product of the full row that's circled.

	I
D	x^2
+ $\ln x$	$\frac{1}{3} x^3$
- $\frac{1}{x}$	

Here's what you write:

$$+\ln x \left(\frac{1}{3} x^3 \right)$$

5. Add the *integral* of the two lowest diagonally adjacent boxes that are circled.

	I
D	x^2
+ $\ln x$	$\frac{1}{3} x^3$
- $\frac{1}{x}$	

Here's what you write:

$$(+\ln x)\left(\frac{1}{3}x^3\right) + \int\left(-\frac{1}{x}\right)\left(\frac{1}{3}x^3\right)dx$$

At this point, you can simplify the first term and integrate the second term:

$$\begin{aligned} &= \frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2 dx \\ &= \frac{1}{3}x^3 \ln x - \left(\frac{1}{3}\right)\left(\frac{1}{3}x^3\right) + C \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C \end{aligned}$$

You can verify this answer by differentiating by using the Product Rule:

$$\begin{aligned} &\frac{d}{dx}\left(\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C\right) \\ &= \frac{1}{3}\left(3x^2 \ln x + \frac{1}{x} \cdot x^3\right) - \frac{1}{3}x^2 \\ &= x^2 \ln x + \frac{1}{3}x^2 - \frac{1}{3}x^2 \\ &= x^2 \ln x \end{aligned}$$

Therefore, this is the correct answer:

$$\int x^2 \ln x dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

1 is for inverse trig

You can integrate four of the six inverse trig functions (arcsin x , arccos x , arctan x , and arccot x) by using the DI-agonal method. By the way, if you haven't memorized the derivatives of the six inverse trig functions (which I give you in Chapter 2), this would be a great time to do so.



Whenever you integrate a product that includes an inverse trig function, this function always goes in the D column.

For example, suppose that you want to integrate

$$\int \arccos x dx$$

1. Write the inverse trig function in the box below the D and the rest of the function value (omitting the dx) in the box below the I .

	I
D	1
+ arccos x	
—	

Note that a 1 goes into the I column.

2. Differentiate arccos x and place the answer in the D column, and then integrate 1 and place the answer in the I column.

	I
D	1
+ arccos x	x
$-\left(-\frac{1}{\sqrt{1-x^2}}\right)$	

3. Add the product of the full row that's circled.

	I
D	1
+ arccos x	x
$-\left(-\frac{1}{\sqrt{1-x^2}}\right)$	

Here's what you write:

$$(+\arccos x)(x)$$

4. Add the integral of the lowest diagonal that's circled.

	I
D	1
+ arccos x	x
$-\left(-\frac{1}{\sqrt{1-x^2}}\right)$	

Here's what you write:

$$(+\arccos x)(x) + \int -\left(-\frac{1}{\sqrt{1-x^2}}\right)(x) dx$$

Simplify and integrate:

$$=x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx$$

$$\text{Let } u = 1 - x^2$$

$$du = -2x dx$$

$$-\frac{1}{2} du = x dx$$



This variable substitution introduces a new variable u . Don't confuse this u with the u used for integration by parts.

$$=x \arccos x + \int \frac{1}{\sqrt{u}} \left(-\frac{1}{2} du\right)$$

$$=x \arccos x - \frac{1}{2}(2\sqrt{u}) + C$$

$$=x \arccos x - \sqrt{u} + C$$

Substituting $1 - x^2$ for u and simplifying gives you this answer:

$$=x \arccos x - \sqrt{1-x^2} + C$$

Therefore, $\int \arccos x dx = x \arccos x - \sqrt{1-x^2} + C$.

A is for algebraic

If you're a bit skeptical that the DI-agonal method is really worth the trouble, I guarantee you that you'll find it useful when handling algebraic factors.

For example, suppose that you want to integrate the following:

$$\int x^3 \sin x \, dx$$

This example is a product of functions, so integration by parts is an option. Going down the LIAT checklist, you notice that the product doesn't contain a log factor or an inverse trig factor. But it does include the algebraic factor x^3 , so place this factor in the D column and the rest in the I column. By now, you're probably getting good at using the chart, so I fill it in for you here.

	I
D	$\sin x$
$+ x^3$	$- \cos x$
$- 3x^2$	

Your next step is normally to write the following

$$+(x^3)(-\cos x) + \int (-3x^2)(-\cos x) \, dx$$

But here comes trouble: The only way to calculate the new integral is by doing *another* integration by parts. And, peeking ahead a bit, here's what you have to look forward to:

$$\begin{aligned} &= (x^3)(-\cos x) - \left[(3x^2)(-\sin x) - \int (6x)(-\sin x) \, dx \right] \\ &= (x^3)(-\cos x) - \left\{ (3x^2)(-\sin x) - \left[(6x)(\cos x) - \int 6 \cos x \, dx \right] \right\} \end{aligned}$$

At last, after integrating by parts *three times*, you finally have an integral that you can solve directly. If evaluating this expression looks like fun (and if you think you can do it quickly on an exam without dropping a minus sign along the way!), by all means go for it. If not, I show you a better way. Read on.



To integrate an algebraic function multiplied by a sine, a cosine, or an exponential function, place the algebraic factor in the D column and the other factor in the I column. Differentiate the algebraic factor down to zero, and then integrate the other factor the same number of times. You can then copy the answer directly from the chart.

Simply extend the DI chart as I show you here.

	I
D	$\sin x$
+ x^3	- $\cos x$
- $3x^2$	- $\sin x$
+ $6x$	$\cos x$
- 6	$\sin x$
+ 0	

Notice that you just continue the patterns in both columns. In the D column, continue alternating plus and minus signs and differentiate until you reach 0. And in the I column, continue integrating.

The very pleasant surprise is that you can now copy the answer from the chart. This answer contains four terms (+ C , of course), which I copy directly from the four circled rows in the chart:

$$x^3 (-\cos x) - 3x^2 (-\sin x) + 6x (\cos x) - 6 (\sin x) + C$$

But wait! Didn't I forget the final integral on the diagonal? Actually, no — but this integral is $\int 0 dx \cdot \sin x = C$, which explains where that final C comes from.

Here's another example, just to show you again how easy the DI-agonal method is for products with algebraic factors:

$$\int 3x^5 e^{2x} dx$$

Without the DI chart, this problem is one gigantic miscalculation waiting to happen. But the chart keeps track of everything.

D	I
	e^{2x}
$+ 3x^5$	$\frac{1}{2} e^{2x}$
$- 15x^4$	$\frac{1}{4} e^{2x}$
$+ 60x^3$	$\frac{1}{8} e^{2x}$
$- 180x^2$	$\frac{1}{16} e^{2x}$
$+ 360x$	$\frac{1}{32} e^{2x}$
$- 360$	$\frac{1}{64} e^{2x}$
$+ 0$	

Now, just copy from the chart, add C , and simplify:

$$\begin{aligned}
 &= +(3x^5)\left(\frac{1}{2}e^{2x}\right) - (15x^4)\left(\frac{1}{4}e^{2x}\right) + (60x^3)\left(\frac{1}{8}e^{2x}\right) - (180x^2)\left(\frac{1}{16}e^{2x}\right) \\
 &\quad + (360x)\left(\frac{1}{32}e^{2x}\right) - (360)\left(\frac{1}{64}e^{2x}\right) + C \\
 &= \frac{3}{2}x^5e^{2x} - \frac{15}{4}x^4e^{2x} + \frac{15}{2}x^3e^{2x} - \frac{45}{4}x^2e^{2x} + \frac{45}{4}xe^{2x} - \frac{45}{8}e^{2x} + C
 \end{aligned}$$

This answer is perfectly acceptable, but if you want to get fancy, factor out $\frac{3}{8}e^{2x}$ and leave a reduced polynomial:

$$= \frac{3}{8}e^{2x}(4x^5 - 10x^4 + 20x^3 - 30x^2 + 30x - 15) + C$$

T is for trig

You can use the DI-agonal method to integrate the product of either a sine or a cosine and an exponential. For example, suppose that you want to evaluate the following integral:

$$\int e^{\frac{x}{3}} \sin x \, dx$$



When integrating either a sine or cosine function multiplied by an exponential function, make your DI-agonal chart with five rows rather than four. Then place the trig function in the D column and the exponential in the I column.

	I
D	$e^{\frac{x}{3}}$
$+ \sin x$	$3e^{\frac{x}{3}}$
$- \cos x$	$9e^{\frac{x}{3}}$
$+(-\sin x)$	

This time, you have two rows to add, as well as the integral of the product of the lowest diagonal:

$$(\sin x)\left(3e^{\frac{x}{3}}\right) + (-\cos x)\left(9e^{\frac{x}{3}}\right) + \int (-\sin x)\left(9e^{\frac{x}{3}}\right) dx$$

This may seem like a dead end because the resulting integral looks so similar to the one that you're trying to evaluate. Oddly enough, however, this similarity makes solving the integral possible. In fact, the next step is to make the integral that results look *exactly* like the one you're trying to solve:

$$= (\sin x)\left(3e^{\frac{x}{3}}\right) + (-\cos x)\left(9e^{\frac{x}{3}}\right) - 9 \int e^{\frac{x}{3}} \sin x dx$$

Next, substitute the variable I for the integral that you're trying to solve. This action isn't strictly necessary, but it makes the course of action a little clearer.

$$I = (\sin x)\left(3e^{\frac{x}{3}}\right) + (-\cos x)\left(9e^{\frac{x}{3}}\right) - 9I$$

Now solve for I using a little basic algebra:

$$10I = (\sin x)\left(3e^{\frac{x}{3}}\right) + (-\cos x)\left(9e^{\frac{x}{3}}\right)$$

$$I = \frac{(\sin x)\left(3e^{\frac{x}{3}}\right) + (-\cos x)\left(9e^{\frac{x}{3}}\right)}{10}$$

Finally, substitute the original integral back into the equation, and add C :

$$\int e^{\frac{x}{3}} \sin x dx = \frac{1}{10} \left[(\sin x)\left(3e^{\frac{x}{3}}\right) + (-\cos x)\left(9e^{\frac{x}{3}}\right) \right] + C$$

Optionally, you can clean up this answer a bit by factoring:

$$\int e^{\frac{x}{3}} \sin x dx = \frac{3}{10} e^{\frac{x}{3}} (\sin x - 3 \cos x) + C$$

If you're skeptical that this method really gives you the right answer, check it by differentiating by using the Product Rule:

$$\begin{aligned} & \frac{d}{dx} \left(\frac{3}{10} e^{\frac{x}{3}} (\sin x - 3 \cos x) + C \right) \\ &= \frac{3}{10} \left[\frac{d}{dx} e^{\frac{x}{3}} (\sin x - 3 \cos x) + \frac{d}{dx} (\sin x - 3 \cos x) \left(e^{\frac{x}{3}} \right) \right] \\ &= \frac{3}{10} \left[\left(\frac{1}{3} e^{\frac{x}{3}} \right) (\sin x - 3 \cos x) + (\cos x + 3 \sin x) \left(e^{\frac{x}{3}} \right) \right] \end{aligned}$$

At this point, algebra shows that this expression is equivalent to the original function:

$$\begin{aligned} &= \frac{1}{10} \left(e^{\frac{x}{3}} \right) (\sin x - 3 \cos x) + \frac{3}{10} (\cos x + 3 \sin x) \left(e^{\frac{x}{3}} \right) \\ &= \frac{1}{10} e^{\frac{x}{3}} \sin x - \frac{3}{10} e^{\frac{x}{3}} \cos x + \frac{3}{10} e^{\frac{x}{3}} \cos x + \frac{9}{10} e^{\frac{x}{3}} \sin x \\ &= \frac{1}{10} e^{\frac{x}{3}} \sin x + \frac{9}{10} e^{\frac{x}{3}} \sin x \\ &= e^{\frac{x}{3}} \sin x \end{aligned}$$

Chapter 7

Trig Substitution: Knowing All the (Tri)Angles

In This Chapter

- ▶ Memorizing the basic trig integrals
 - ▶ Integrating powers of sines and cosines, tangents and secants, and cotangents and cosecants
 - ▶ Understanding the three cases for using trig substitution
 - ▶ Avoiding trig substitution when possible
-

Trig substitution is another technique to throw in your ever-expanding bag of integration tricks. It allows you to integrate functions that contain radicals of polynomials such as $\sqrt{4-x^2}$ and other similar difficult functions.

Trig substitution may remind you of variable substitution, which I discuss in Chapter 5. With both types of substitution, you break the function that you want to integrate into pieces and express each piece in terms of a new variable. With trig substitution, however, you express these pieces as trig functions.

So, before you can do trig substitution, you need to be able to integrate a wider variety of products and powers of trig functions. The first few parts of this chapter give you the skills that you need. After that, I show you how to use trig substitution to express very complicated-looking radical functions in terms of trig functions.

Integrating the Six Trig Functions

You already know how to integrate $\sin x$ and $\cos x$ from Chapter 4, but for completeness, here are the integrals of all six trig functions:

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

Please commit these to memory — you need them! For practice, you can also try differentiating each result to show why each of these integrals is correct.

Integrating Powers of Sines and Cosines

Later in this chapter, when I show you trig substitution, you need to know how to integrate powers of sines and cosines in a variety of combinations. In this section, I show you what you need to know.

Odd powers of sines and cosines

You can integrate *any* function of the form $\sin^m x \cos^n x$ when m is odd, for any real value of n . For this procedure, keep in mind the handy trig identity $\sin^2 x + \cos^2 x = 1$. For example, here's how you integrate $\sin^7 x \cos^{\frac{1}{3}} x$:

- 1. Peel off a $\sin x$ and place it next to the dx :**

$$\int \sin^7 x \cos^{\frac{1}{3}} x \, dx = \int \sin^6 x \cos^{\frac{1}{3}} x \sin x \, dx$$

- 2. Apply the trig identity $\sin^2 x = 1 - \cos^2 x$ to express the rest of the sines in the function as cosines:**

$$= \int (1 - \cos^2 x)^3 \cos^{\frac{1}{3}} x \sin x \, dx$$

- 3. Use the variable substitution $u = \cos x$ and $du = -\sin x \, dx$:**

$$= -\int (1 - u^2)^3 u^{\frac{1}{3}} \, du$$

Now that you have the function in terms of powers of u , the worst is over. You can expand the function out, turning it into a polynomial. This is just algebra:

$$\begin{aligned} &= - \int (1 - u^2)(1 - u^2)(1 - u^2) u^{\frac{1}{3}} du \\ &= - \int (1 - 3u^2 + 3u^4 - u^6) u^{\frac{1}{3}} du \\ &= - \int \left(u^{\frac{1}{3}} - 3u^{\frac{7}{3}} + 3u^{\frac{13}{3}} - u^{\frac{19}{3}} \right) du \end{aligned}$$

To continue, use the Sum Rule and Constant Multiple Rule to separate this into four integrals, as I show you in Chapter 4. Don't forget to distribute that minus sign to all four integrals!

$$= - \int u^{\frac{1}{3}} du + 3 \int u^{\frac{7}{3}} du - 3 \int u^{\frac{13}{3}} du + \int u^{\frac{19}{3}} du$$

At this point, you can evaluate each integral separately by using the Power Rule:

$$= \frac{3}{4} u^{\frac{4}{3}} + \frac{9}{10} u^{\frac{10}{3}} - \frac{9}{16} u^{\frac{16}{3}} + \frac{3}{22} u^{\frac{22}{3}} + C$$

Finally, use $u = \cos x$ to reverse the variable substitution:

$$= \frac{3}{4} \cos^{\frac{4}{3}} x + \frac{9}{10} \cos^{\frac{10}{3}} x - \frac{9}{16} \cos^{\frac{16}{3}} x + \frac{3}{22} \cos^{\frac{22}{3}} x + C$$

Notice that when you substitute back in terms of x , the power goes next to the \cos rather than the x , because you're raising the entire function $\cos x$ to a power. (See Chapter 2 if you're unclear about this point.)

Similarly, you integrate *any* function of the form $\sin^m x \cos^n x$ when n is odd, for any real value of m . These steps are practically the same as those in the previous example. For example, here's how you integrate $\sin^{-4} x \cos^9 x$:

1. Peel off a $\cos x$ and place it next to the dx :

$$\int \sin^{-4} x \cos^9 x dx = \int \sin^{-4} x \cos^8 x \cos x dx$$

2. Apply the trig identity $\cos^2 x = 1 - \sin^2 x$ to express the rest of the cosines in the function as sines:

$$= \int \sin^{-4} x (1 - \sin^2 x)^4 \cos x dx$$

3. Use the variable substitution $u = \sin x$ and $du = \cos x dx$:

$$= \int u^{-4} (1 - u^2)^4 du$$

At this point, you can distribute the function to turn it into a polynomial and then integrate it as I show you in the previous example.

Even powers of sines and cosines

To integrate $\sin^2 x$ and $\cos^2 x$, use the two half-angle trig identities that I show you in Chapter 2:

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

For example, here's how you integrate $\cos^2 x$:

- 1. Use the half-angle identity for cosine to rewrite the integral in terms of $\cos 2x$:**

$$\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx$$

- 2. Use the Constant Multiple Rule to move the denominator outside the integral:**

$$= \frac{1}{2} \int (1 + \cos 2x) \, dx$$

- 3. Distribute the function and use the Sum Rule to split it into several integrals:**

$$= \frac{1}{2} \left(\int 1 \, dx + \int \cos 2x \, dx \right)$$

- 4. Evaluate the two integrals separately:**

$$= \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) + C$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x + C$$

As a second example, here's how you integrate $\sin^2 x \cos^4 x$:

- 1. Use the two half-angle identities to rewrite the integral in terms of $\cos 2x$:**

$$\int \sin^2 x \cos^4 x \, dx = \int \frac{1 - \cos 2x}{2} \left(\frac{1 + \cos 2x}{2} \right)^2 \, dx$$

- 2. Use the Constant Multiple Rule to move the denominators outside the integral:**

$$= \frac{1}{8} \int (1 - \cos 2x)(1 + \cos 2x)^2 \, dx$$

- 3. Distribute the function and use the Sum Rule to split it into several integrals:**

$$= \frac{1}{8} \left(\int 1 \, dx + \int \cos 2x \, dx - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx \right)$$

- Evaluate the resulting odd-powered integrals by using the procedure from the earlier section “Odd powers of sines and cosines,” and evaluate the even-powered integrals by returning to Step 1 of the previous example.

Integrating Powers of Tangents and Secants

When you’re integrating powers of tangents and secants, here’s the rule to remember: *Eeven* powers of *seeecants* are *eeeasy*. The three *E*s in the *keeyy* words should help you remember this rule. By the way, odd powers of tangents are also easy. You’re on your own remembering this fact!

In this section, I show you how to integrate $\tan^m x \sec^n x$ for all positive integer values of m and n . You use this skill later in this chapter, when I show you how to do trig substitution.

Even powers of secants with tangents

To integrate $\tan^m x \sec^n x$ when n is even — for example, $\tan^8 x \sec^6 x$ — follow these steps:

- Peel off a $\sec^2 x$ and place it next to the dx :

$$\int \tan^8 x \sec^6 x \, dx = \int \tan^8 x \sec^4 x \sec^2 x \, dx$$

- Use the trig identity $1 + \tan^2 x = \sec^2 x$ to express the remaining secant factors in terms of tangents:

$$= \int \tan^8 x (1 + \tan^2 x)^2 \sec^2 x \, dx$$

- Use the variable substitution $u = \tan x$ and $du = \sec^2 x \, dx$:

$$\int u^8 (1 + u^2)^2 \, du$$

At this point, the integral is a polynomial, and you can evaluate it as I show you in Chapter 4.

Odd powers of tangents with secants

To integrate $\tan^m x \sec^n x$ when m is odd — for example, $\tan^7 x \sec^9 x$ — follow these steps:

1. Peel off a $\tan x$ and a $\sec x$ and place them next to the dx :

$$\int \tan^7 x \sec^9 x = \int \tan^6 x \sec^8 x \sec x \tan x dx$$

2. Use the trig identity $\tan^2 x = \sec^2 x - 1$ to express the remaining tangent factors in terms of secants:

$$= \int (\sec^2 x - 1)^3 \sec^8 x \sec x \tan x dx$$

3. Use the variable substitution $u = \sec x$ and $du = \sec x \tan x dx$:

$$= \int (u^2 - 1)^3 u^8 du$$

At this point, the integral is a polynomial, and you can evaluate it as I show you in Chapter 4.

Odd powers of tangents without secants

To integrate $\tan^m x$ when m is odd, use a trig identity to convert the function to sines and cosines as follows:

$$\int \tan^m x dx = \int \frac{\sin^m x}{\cos^m x} dx = \int \sin^m x \cos^{-m} x dx$$

After that, you can integrate by using the procedure from the earlier section, “Odd powers of sines and cosines.”

Even powers of tangents without secants

To integrate $\tan^m x$ when m is even — for example, $\tan^8 x$ — follow these steps:

1. Peel off a $\tan^2 x$ and use the trig identity $\tan^2 x = \sec^2 x - 1$ to express it in terms of $\tan x$:

$$\int \tan^8 x dx = \int \tan^6 x (\sec^2 x - 1) dx$$

2. Distribute to split the integral into two separate integrals:

$$= \int \tan^6 x \sec^2 x dx - \int \tan^6 x dx$$

3. Evaluate the first integrals using the procedure I show you in the earlier section “Even powers of secants with tangents.”
4. Return to Step 1 to evaluate the second integral.

Even powers of secants without tangents

To integrate $\sec^n x$ when n is even — for example, $\sec^4 x$ — follow these steps:

1. Use the trig identity $1 + \tan^2 x = \sec^2 x$ to express the function in terms of tangents:

$$\int \sec^4 x \, dx = \int (1 + \tan^2 x)^2 \, dx$$

2. Distribute and split the integral into three or more integrals:

$$= \int 1 \, dx + 2 \int \tan^2 x \, dx + \int \tan^4 x \, dx$$

3. Integrate all powers of tangents by using the procedures from the sections on powers of tangents without secants.

Odd powers of secants without tangents

This is the hardest case, so fasten your seat belt. To integrate $\sec^n x$ when n is odd — for example, $\sec^3 x$ — follow these steps:

1. Peel off a sec x :

$$\int \sec^3 x \, dx = \int \sec^2 x \sec x \, dx$$

2. Use the trig identity $1 + \tan^2 x = \sec^2 x$ to express the remaining secants in terms of tangents:

$$= \int (1 + \tan^2 x) \sec x \, dx$$

3. Distribute and split the integral into two or more integrals:

$$= \int \sec x + \int \tan^2 x \sec x \, dx$$

4. Evaluate the first integral:

$$= \ln|\sec x + \tan x| + \int \tan^2 x \sec x \, dx$$

You can omit the constant C because you still have an integral that you haven't evaluated yet — just don't forget to put it in at the end.

5. Integrate the second integral by parts by differentiating $\tan x$ and integrating $\sec x \tan x$ (see Chapter 6 for more on integration by parts):

$$= \ln|\sec x + \tan x| + \tan x \sec x - \int \sec^3 x \, dx$$

At this point, notice that you've shown the following equation to be true:

$$\int \sec^3 x \, dx = \ln|\sec x + \tan x| + \tan x \sec x - \int \sec^3 x \, dx$$

6. Follow the algebraic procedure that I outline in Chapter 6.

First, substitute the variable I for the integral on both sides of the equation:

$$I = \ln|\sec x + \tan x| + \tan x \sec x - I$$

Now, solve this equation for I :

$$2I = \ln|\sec x + \tan x| + \tan x \sec x$$

$$I = \frac{1}{2} \ln|\sec x + \tan x| + \frac{1}{2} \tan x \sec x$$

Now, you can substitute the integral back for I . Don't forget, however, that you need to add a constant to the right side of this equation, to cover all possible solutions to the integral:

$$\int \sec^3 x \, dx = \frac{1}{2} \ln|\sec x + \tan x| + \frac{1}{2} \tan x \sec x + C$$

That's your final answer. I truly hope that you never have to integrate $\sec^5 x$, let alone higher odd powers of a secant. But if you do, the basic procedure I outline here will provide you with a value for $\int \sec^5 x \, dx$ in terms of $\int \sec^3 x \, dx$. Good luck!

Even powers of tangents with odd powers of secants

To integrate $\tan^m x \sec^n x$ when m is even and n is odd, transform the function into an odd power of secants, and then use the method that I outline in the previous section "Odd powers of secants without tangents."

For example, here's how you integrate $\tan^4 x \sec^3 x$:

1. Use the trusty trig identity $\tan^2 x = \sec^2 x - 1$ to convert all the tangents to secants:

$$\int \tan^4 x \sec^3 x \, dx = \int (\sec^2 x - 1)^2 \sec^3 x \, dx$$

2. Distribute the function and split the integral by using the Sum Rule:

$$= \int \sec^7 x \, dx - \int 2 \sec^5 x \, dx + \int \sec^3 x \, dx$$

3. Solve the resulting odd-powered integrals by using the procedure from “Odd powers of secants without tangents.”

Unfortunately, this procedure brings you back to the most difficult case in this section. Fortunately, most teachers are fairly merciful when you’re working with these functions, so you probably won’t have to face this integral on an exam. If you do, however, you have my deepest sympathy.

Integrating Powers of Cotangents and Cosecants

The methods for integrating powers of cotangents and cosecants are very close to those for tangents and secants, which I show you in the preceding section. For example, in the earlier section “Even powers of secants with tangents,” I show you how to integrate $\tan^8 x \sec^6 x$. Here’s how to integrate $\cot^8 x \csc^6 x$:

1. Peel off a $\csc^2 x$ and place it next to the dx :

$$\int \cot^8 x \csc^6 x \, dx = \int \cot^8 x \csc^4 x \csc^2 x \, dx$$

2. Use the trig identity $1 + \cot^2 x = \csc^2 x$ to express the remaining cosecant factors in terms of cotangents:

$$= \int \cot^8 x (1 + \cot^2 x)^2 \csc^2 x \, dx$$

3. Use the variable substitution $u = \cot x$ and $du = -\csc^2 x \, dx$:

$$= - \int u^8 (1 + u^2)^2 \, du$$

At this point, the integral is a polynomial, and you can evaluate it as I show you in Chapter 4.

Notice that the steps here are virtually identical to those for tangents and secants. The biggest change here is the introduction of a minus sign in Step 3. So, to find out everything you need to know about integrating cotangents and cosecants, try all the examples in the previous section, but switch every tangent to a cotangent and every secant to a cosecant.



Sometimes, knowing how to integrate cotangents and cosecants can be useful for integrating negative powers of other trig functions — that is, powers of trig functions in the denominator of a fraction.

For example, suppose that you want to integrate $\frac{\cos^2 x}{-\sin^6 x}$. The methods that I outline earlier don't work very well in this case, but you can use trig identities to express it as cotangents and cosecants.

$$\frac{\cos^2 x}{\sin^6 x} = \frac{\cos^2 x}{\sin^2 x} \cdot \frac{1}{\sin^4 x} = \cot^2 x \csc^4 x$$

I show you more about this in the next section “Integrating Weird Combinations of Trig Functions.”

Integrating Weird Combinations of Trig Functions

You don't really have to know how to integrate *every* possible trig function to pass Calculus II. If you can do all the techniques that I introduce earlier in this chapter — and I admit that's a lot to ask! — then you'll be able to handle most of what your professor throws at you with ease. You'll also have a good shot at hitting any curveballs that come at you on an exam.

But in case you're nervous about the exam and would rather study than worry, in this section I show you how to integrate a wider variety of trig functions. I don't promise to cover *all* possible trig functions exhaustively. But I do give you a few additional ways to think about and categorize trig functions that could help you when you're in unfamiliar territory.

Using identities to tweak functions

You can express every product of powers of trig functions, no matter how weird, as the product of any pair of trig functions. The three most useful pairings (as you may guess from earlier in this chapter) are sine and cosine, tangent and secant, and cotangent and cosecant. Table 7-1 shows you how to express all six trig functions as each of these pairings.

Table 7-1 Expressing the Six Trig Functions
As a Pair of Trig Functions

<i>Trig Function</i>	<i>As Sines & Cosines</i>	<i>As Tangents & Secants</i>	<i>As Cotangents & Cosecants</i>
$\sin x$	$\sin x$	$\frac{\tan x}{\sec x}$	$\frac{1}{\csc x}$
$\cos x$	$\cos x$	$\frac{1}{\sec x}$	$\frac{\cot x}{\csc x}$
$\tan x$	$\frac{\sin x}{\cos x}$	$\tan x$	$\frac{1}{\cot x}$
$\cot x$	$\frac{\cos x}{\sin x}$	$\frac{1}{\tan x}$	$\cot x$
$\sec x$	$\frac{1}{\cos x}$	$\sec x$	$\frac{\csc x}{\cot x}$
$\csc x$	$\frac{1}{\sin x}$	$\frac{\sec x}{\tan x}$	$\csc x$

For example, look at the following function:

$$\frac{\cos x \cot^3 x \csc^2 x}{\sin^2 x \tan x \sec x}$$

As it stands, you can't do much to integrate this monster. But try expressing it in terms of each possible pairing of trig functions:

$$\begin{aligned} &= \frac{\cos^6 x}{\sin^8 x} \\ &= \frac{\sec^2 x}{\tan^8 x} \\ &= \cot^6 x \csc^2 x \end{aligned}$$

As it turns out, the most useful pairing for integration in this case is $\cot^6 x \csc^2 x$. No fraction is present — that is, both terms are raised to positive powers — and the cosecant term is raised to an even power, so you can use the same basic procedure that I show you in the earlier section “Even powers of secants with tangents.”

Using Trig Substitution

Trig substitution is similar to variable substitution (which I discuss in Chapter 5), using a change in variable to turn a function that you can't integrate into one that you can. With variable substitution, you typically use the variable u . With trig substitution, however, you typically use the variable θ .

Trig substitution allows you to integrate a whole slew of functions that you can't integrate otherwise. These functions have a special, uniquely scary look about them, and are variations on these three themes:

$$(a^2 - bx^2)^n$$

$$(a^2 + bx^2)^n$$

$$(bx^2 - a^2)^n$$



Trig substitution is most useful when n is $\frac{1}{2}$ or a negative number — that is, for hairy square roots and polynomials in the denominator of a fraction. When n is a positive integer, your best bet is to express the function as a polynomial and integrate it as I show you in Chapter 4.

In this section, I show you how to use trig substitution to integrate functions like these. But, before you begin, take this simple test:

Trig substitution is:

- ✓ A) Easy and *fun* — even a child can do it!
- ✓ B) Not so bad when you know how.
- ✓ C) About as attractive as drinking bleach.

I wish I could tell you that the answer is A, but then I'd be a big liarmouth and you'd never trust me again. So I admit that trig substitution is less fun than a toga party with a hot date. At the same time, your worst trig substitution nightmares don't have to come true, so please put the bottle of bleach back in the laundry room.

I have the system right here, and if you follow along closely, I give you the tool that you need to make trig substitution mostly a matter of filling in the blanks. Trust me — have I ever lied to you?

Distinguishing three cases for trig substitution

Trig substitution is useful for integrating functions that contain three very recognizable types of polynomials in either the numerator or denominator. Table 7-2 lists the three cases that you need to know about.

Table 7-2 The Three Trig Substitution Cases		
Case	Radical of Polynomial	Example
Sine case	$(a^2 - bx^2)^n$	$\int \sqrt{4 - x^2} dx$
Tangent case	$(a^2 + bx^2)^n$	$\int \frac{1}{(4 + 9x^2)^2} dx$
Secant case	$(bx^2 - a^2)^n$	$\int \frac{1}{\sqrt{16x^2 - 1}} dx$

The first step to trig substitution is being able to recognize and distinguish these three cases when you see them.



Knowing the formulas for differentiating the inverse trig functions can help you remember these cases.

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{x\sqrt{x^2 - 1}}$$

Note that the differentiation formula for $\arcsin x$ contains a polynomial that looks like the sine case: a constant minus x^2 . The formula for $\arctan x$ contains a polynomial that looks like the tangent case: a constant plus x^2 . And the formula for $\operatorname{arcsec} x$ contains a polynomial that looks like the secant case: x^2 minus a constant. So, if you already know these formulas, you don't have to memorize any additional information.

Integrating the three cases

Trig substitution is a five-step process:

1. Draw the trig substitution triangle for the correct case.
2. Identify the separate pieces of the integral (including dx) that you need to express in terms of θ .
3. Express these pieces in terms of trig functions of θ .
4. Rewrite the integral in terms of θ and evaluate it.
5. Substitute x for θ in the result.

Don't worry if these steps don't make much sense yet. In this section, I show you how to do trig substitution for each of the three cases.

The sine case

When the function you're integrating includes a term of the form $(a^2 - bx^2)^n$, draw your trig substitution triangle for the *sine case*. For example, suppose that you want to evaluate the following integral:

$$\int \sqrt{4 - x^2} dx$$

This is a *sine case*, because a constant minus a multiple of x^2 is being raised to a power $\left(\frac{1}{2}\right)$. Here's how you use trig substitution to handle the job:

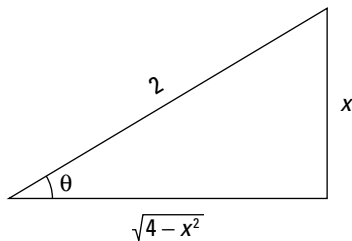
1. Draw the trig substitution triangle for the correct case.

Figure 7-1 shows you how to fill in the triangle for the sine case. Notice that the radical goes on the *adjacent* side of the triangle. Then, to fill in the other two sides of the triangle, I use the square roots of the two terms inside the radical — that is, 2 and x . I place 2 on the hypotenuse and x on the opposite side.

You can check to make sure that this placement is correct by using the Pythagorean theorem: $x^2 + (\sqrt{4 - x^2})^2 = 2^2$.



Figure 7-1:
A trig substitution triangle for the sine case.



2. Identify the separate pieces of the integral (including dx) that you need to express in terms of θ .

In this case, the function contains two separate pieces that contain x : $\sqrt{4 - x^2}$ and dx .

3. Express these pieces in terms of trig functions of θ .

This is the real work of trig substitution, but when your triangle is set up properly, this work becomes a lot easier. In the sine case, *all* trig functions should be sines and cosines.

To represent the radical portion as a trig function of θ , first build a fraction using the radical $\sqrt{4-x^2}$ as the numerator and the constant 2 as the denominator. Then set this fraction equal to the appropriate trig function:

$$\frac{\sqrt{4-x^2}}{2} = \cos \theta$$

Because the numerator is the adjacent side of the triangle and the denominator is the hypotenuse $\left(\frac{A}{H}\right)$, this fraction is equal to $\cos \theta$. Now, a little algebra gets the radical alone on one side of the equation:

$$\sqrt{4-x^2} = 2 \cos \theta$$

Next, you want to express dx as a trig function of θ . To do so, build another fraction with the variable x in the numerator and the constant 2 in the denominator. Then set this fraction equal to the correct trig function:

$$\frac{x}{2} = \sin \theta$$

This time, the numerator is the opposite side of the triangle and the denominator is the hypotenuse $\left(\frac{O}{H}\right)$, so this fraction is equal to $\sin \theta$. Now, solve for x and then differentiate:

$$x = 2 \sin \theta$$

$$dx = 2 \cos \theta d\theta$$

4. Rewrite the integral in terms of θ and evaluate it:

$$\begin{aligned} & \int \sqrt{4-x^2} dx \\ & \int 2 \cos \theta \cdot 2 \cos \theta d\theta \\ & = 4 \int \cos^2 \theta d\theta \end{aligned}$$

Knowing how to evaluate trig integrals really pays off here. I cut to the chase in this example, but earlier in this chapter (in “Integrating Powers of Sines and Cosines”), I show you how to integrate all sorts of trig functions like this one:

$$= 2\theta + \sin 2\theta + C$$

5. To change those two θ terms into x terms, reuse the following equation:

$$\begin{aligned} \frac{x}{2} &= \sin \theta \\ \theta &= \arcsin \frac{x}{2} \end{aligned}$$

So here's a substitution that gives you an answer:

$$= 2 \arcsin \frac{x}{2} + \sin(2 \arcsin \frac{x}{2}) + C$$

This answer is perfectly valid so, technically speaking, you can stop here. However, some professors frown upon the nesting of trig and inverse trig functions, so they'll prefer a simplified version of $\sin(2 \arcsin \frac{x}{2})$. To find this, start by applying the double-angle sine formula (see Chapter 2) to $\sin 2\theta$:

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

Now, use your trig substitution triangle to substitute values for $\sin \theta$ and $\cos \theta$ in terms of x :

$$\begin{aligned} &= 2 \left(\frac{x}{2} \right) \left(\frac{\sqrt{4-x^2}}{2} \right) \\ &= \frac{1}{2} x \sqrt{4-x^2} \end{aligned}$$

To finish up, substitute this expression for that problematic second term to get your final answer in a simplified form:

$$\begin{aligned} &2\theta + \sin 2\theta + C \\ &= 2 \arcsin \frac{x}{2} + \frac{1}{2} x \sqrt{4-x^2} + C \end{aligned}$$

The tangent case

When the function you're integrating includes a term of the form $(a^2 + x^2)^n$, draw your trig substitution triangle for the *tangent case*. For example, suppose that you want to evaluate the following integral:

$$\int \frac{1}{(4+9x^2)^2} dx$$

This is a tangent case, because a constant plus a multiple of x^2 is being raised to a power (-2). Here's how you use trig substitution to integrate:

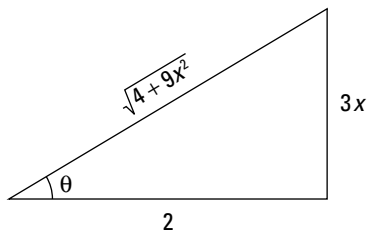
1. Draw the trig substitution triangle for the tangent case.

Figure 7-2 shows you how to fill in the triangle for the tangent case. Notice that the radical of what's inside the parentheses goes on the *hypotenuse* of the triangle. Then, to fill in the other two sides of the triangle, use the square roots of the two terms inside the radical — that is, 2 and $3x$. Place the constant term 2 on the adjacent side and the variable term $3x$ on the opposite side.



With the tangent case, make sure not to mix up your placement of the variable and the constant.

Figure 7-2:
A trig substitution triangle for the tangent case.



2. Identify the separate pieces of the integral (including dx) that you need to express in terms of θ .

In this case, the function contains two separate pieces that contain x :

$$\frac{1}{(4 + 9x^2)^2} \text{ and } dx.$$

3. Express these pieces in terms of trig functions of θ .

In the tangent case, *all* trig functions should be initially expressed as tangents and secants.

To represent the rational portion as a trig function of θ , build a fraction using the radical $\sqrt{4 + 9x^2}$ as the numerator and the constant 2 as the denominator. Then set this fraction equal to the appropriate trig function:

$$\frac{\sqrt{4 + 9x^2}}{2} = \sec \theta$$

Because this fraction is the hypotenuse of the triangle over the adjacent side ($\frac{H}{A}$), it's equal to $\sec \theta$. Now, use algebra and trig identities to tweak this equation into shape:

$$\begin{aligned} \sqrt{4 + 9x^2} &= 2 \sec \theta \\ (4 + 9x^2)^2 &= 8 \sec^4 \theta \\ \frac{1}{(4 + 9x^2)^2} &= \frac{1}{8 \sec^4 \theta} \end{aligned}$$

Next, express dx as a trig function of θ . To do so, build another fraction with the variable $3x$ in the numerator and the constant 2 in the denominator:

$$\frac{3x}{2} = \tan \theta$$

This time, the fraction is the opposite side of the triangle over the adjacent side $\left(\frac{O}{A}\right)$, so it equals $\tan \theta$. Now, solve for x and then differentiate:

$$x = \frac{2}{3} \tan \theta$$

$$dx = \frac{2}{3} \sec^2 \theta d\theta$$

4. Express the integral in terms of θ and evaluate it:

$$\begin{aligned} \int \frac{1}{(4+9x^2)^2} dx \\ = \int \frac{1}{8\sec^4 \theta} \cdot \frac{2}{3} \sec^2 \theta d\theta \end{aligned}$$

Now, some cancellation and reorganization turns this nasty-looking integral into something manageable:

$$= \frac{1}{12} \int \cos^2 \theta d\theta$$

At this point, use your skills from the earlier section “Even Powers of Sines and Cosines” to evaluate this integral:

$$= \frac{1}{24} \theta + \frac{1}{48} \sin 2\theta + C$$

5. Change the two θ terms back into x terms:

You need to find a way to express θ in terms of x . Here’s the simplest way:

$$\tan \theta = \frac{3x}{2}$$

$$\theta = \arctan \frac{3x}{2}$$

So here’s a substitution that gives you an answer:

$$\frac{1}{24} \theta + \frac{1}{48} \sin 2\theta + C = \frac{1}{24} \arctan \frac{3x}{2} + \frac{1}{48} \sin \left(2 \arctan \frac{3x}{2} \right) + C$$

This answer is valid, but most professors won’t be crazy about that ugly second term, with the sine of an arctangent. To simplify it, apply the double-angle sine formula (see Chapter 2) to $\frac{1}{48} \sin 2\theta$:

$$\frac{1}{48} \sin 2\theta = \frac{1}{24} \sin \theta \cos \theta$$

Now, use your trig substitution triangle to substitute values for $\sin \theta$ and $\cos \theta$ in terms of x :

$$\begin{aligned}
 &= \frac{1}{24} \left(\frac{3x}{\sqrt{4+9x^2}} \right) \left(\frac{2}{\sqrt{4+9x^2}} \right) \\
 &= \frac{6x}{24(4+9x^2)} \\
 &= \frac{x}{(16+36x^2)}
 \end{aligned}$$

Finally, use this result to express the answer in terms of x :

$$\begin{aligned}
 &\frac{1}{24} \theta + \frac{1}{48} \sin 2\theta + C \\
 &= \frac{1}{24} \arctan \frac{3x}{2} + \frac{x}{(16+36x^2)} + C
 \end{aligned}$$

The secant case

When the function that you're integrating includes a term of the form $(bx^2 - a^2)^n$, draw your trig substitution triangle for the *secant case*. For example, suppose that you want to evaluate this integral:

$$\int \frac{1}{\sqrt{16x^2 - 1}} dx$$

This is a secant case, because a multiple of x^2 minus a constant is being raised to a power $\left(-\frac{1}{2}\right)$. Integrate by using trig substitution as follows:

1. Draw the trig substitution triangle for the secant case.

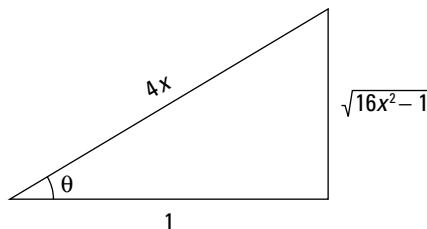
Figure 7-3 shows you how to fill in the triangle for the secant case. Notice that the radical goes on the *opposite* side of the triangle. Then, to fill in the other two sides of the triangle, use the square roots of the two terms inside the radical — that is, 1 and $4x$. Place the constant 1 on the adjacent side and the variable $4x$ on the hypotenuse.

You can check to make sure that this placement is correct by using the Pythagorean theorem: $1^2 + (\sqrt{16x^2 - 1})^2 = (4x)^2$.



Figure 7-3:

A trig substitution triangle for the secant case.



- 2. Identify the separate pieces of the integral (including dx) that you need to express in terms of θ .**

In this case, the function contains two separate pieces that contain x :

$$\frac{1}{\sqrt{16x^2 - 1}} \text{ and } dx.$$

- 3. Express these pieces in terms of trig functions of θ .**

In the secant case (as in the tangent case), *all* trig functions should be initially represented as tangents and secants.

To represent the radical portion as a trig function of θ , build a fraction by using the radical $\sqrt{16x^2 - 1}$ as the numerator and the constant 1 as the denominator. Then set this fraction equal to the appropriate trig function:

$$\frac{\sqrt{16x^2 - 1}}{1} = \tan \theta$$

Notice that this fraction is the opposite side of the triangle over the adjacent side $\left(\frac{O}{A}\right)$, so it equals $\tan \theta$. Simplifying it a bit gives you this equation:

$$\frac{1}{\sqrt{16x^2 - 1}} = \frac{1}{\tan \theta}$$

Next, express dx as a trig function of θ . To do so, build another fraction with the variable x in the numerator and the constant 1 in the denominator:

$$\frac{4x}{1} = \sec \theta$$

This time, the fraction is the hypotenuse over the adjacent side of the triangle $\left(\frac{H}{A}\right)$, which equals $\sec \theta$. Now, solve for x and differentiate to find dx :

$$x = \frac{1}{4} \sec \theta$$

$$dx = \frac{1}{4} \sec \theta \tan \theta d\theta$$

- 4. Express the integral in terms of θ and evaluate it:**

$$\begin{aligned} \int \frac{1}{\sqrt{16x^2 - 1}} dx &= \int \frac{1}{\tan \theta} \cdot \frac{1}{4} \sec \theta \tan \theta d\theta \\ &= \frac{1}{4} \int \sec \theta d\theta \end{aligned}$$

Now, use the formula for the integral of the secant function from “Integrating the Six Trig Functions” earlier in this chapter:

$$= \frac{1}{4} \ln |\sec \theta + \tan \theta| + C$$

5. Change the two θ terms back into x terms:

In this case, you don't have to find the value of θ because you already know the values of $\sec \theta$ and $\tan \theta$ in terms of x from Step 3. So, substitute these two values to get your final answer:

$$= \frac{1}{4} \ln |4x + \sqrt{16x^2 - 1}| + C$$

Knowing when to avoid trig substitution

Now that you know how to use trig substitution, I give you a skill that can be even more valuable: *avoiding* trig substitution when you don't need it. For example, look at the following integral:

$$\int (1 - 4x^2)^2 dx$$

This might look like a good place to use trig substitution, but it's an even better place to use a little algebra to expand the problem into a polynomial:

$$= \int (1 - 8x^2 + 16x^4) dx$$

Similarly, look at this integral:

$$\int \frac{x}{\sqrt{x^2 - 49}} dx$$

You can use trig substitution to evaluate this integral if you want to. (You can also walk to the top of the Empire State Building instead of taking the elevator if that tickles your fancy.) However, the presence of that little x in the numerator should tip you off that variable substitution will work just as well (flip to Chapter 5 for more on variable substitution):

$$\text{Let } u = x^2 - 49$$

$$du = 2x dx$$

$$\frac{1}{2} du = x dx$$

Using this substitution results in the following integral:

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$= \sqrt{u} + C$$

$$= \sqrt{x^2 - 49} + C$$

Done! I probably don't need to tell you how much time and aggravation you can save by working smarter rather than harder. So I won't!

Chapter 8

When All Else Fails: Integration with Partial Fractions

In This Chapter

- ▶ Rewriting complicated fractions as the sum of two or more partial fractions
 - ▶ Knowing how to use partial fractions in four distinct cases
 - ▶ Integrating with partial fractions
 - ▶ Using partial fractions with improper rational expressions
-

Let's face it: At this point in your math career, you have bigger things to worry about than adding a couple of fractions. And if you've survived integration by parts (Chapter 6) and trig integration (Chapter 7), multiplying a few polynomials isn't going to kill you either.

So, here's the good news about partial fractions: They're based on very simple arithmetic and algebra. In this chapter, I introduce you to the basics of partial fractions and show you how to use them to evaluate integrals. I illustrate four separate cases in which partial fractions can help you integrate functions that would otherwise be a big ol' mess.

Now, here's the bad news: Although the concept of partial fractions isn't difficult, using them to integrate is just about the most tedious thing you encounter in this book. And as if that weren't enough, partial fractions only work with *proper* rational functions, so I show you how to distinguish these from their ornery cousins, *improper* rational functions. I also give you a big blast from the past, a refresher on *polynomial division*, which I promise is easier than you remember it to be.

Strange but True: Understanding Partial Fractions

Partial fractions are useful for integrating *rational functions* — that is, functions in which a polynomial is divided by a polynomial. The basic tactic behind partial fractions is to split up a rational function that you can't integrate into two or more simpler functions that you can integrate.

In this section, I show you a simple analogy for partial fractions that involves only arithmetic. After you understand this analogy, partial fractions make a lot more sense. At the end of the section, I show you how to solve an integral by using partial fractions.

Looking at partial fractions

Suppose that you want to split the fraction $\frac{14}{15}$ into a sum of two smaller fractions. Start by decomposing the denominator down to its factors — 3 and 5 — and setting the denominators of these two smaller fractions to these numbers:

$$\frac{14}{15} = \frac{A}{3} + \frac{B}{5} = \frac{5A + 3B}{15}$$

So, you want to find an A and a B that satisfy this equation:

$$5A + 3B = 14$$

Now, just by eyeballing this fraction, you can probably find the nice integer solution $A = 1$ and $B = 3$, so:

$$\frac{14}{15} = \frac{1}{3} + \frac{3}{5}$$

If you include negative fractions, you can find integer solutions like this for every fraction. For example, the fraction $\frac{1}{15}$ seems too small to be a sum of thirds and fifths, until you discover:

$$\frac{2}{3} - \frac{3}{5} = \frac{1}{15}$$

Using partial fractions with rational expressions

The technique of breaking up fractions works for rational expressions. It can provide a strategy for integrating functions that you can't compute directly. For example, suppose that you're trying to find this integral:

$$\int \frac{6}{x^2 - 9} dx$$

You can't integrate this function directly, but if you break it into the sum of two simpler rational expressions, you can use the Sum Rule to solve them separately. And, fortunately, the polynomial in the denominator factors easily:

$$\frac{6}{x^2 - 9} = \frac{6}{(x + 3)(x - 3)}$$

So, set up this polynomial fraction just as I do with the regular fractions in the previous section:

$$\begin{aligned} \frac{6}{(x + 3)(x - 3)} &= \frac{A}{x + 3} + \frac{B}{x - 3} \\ &= \frac{A(x - 3) + B(x + 3)}{(x + 3)(x - 3)} \end{aligned}$$

This gives you the following equation:

$$A(x - 3) + B(x + 3) = 6$$

This equation works for *all* values of x . You can exploit this fact to find the values of A and B by picking helpful values of x . To solve this equation for A and B , substitute the roots of the original polynomial (3 and -3) for x and watch what happens:

$$\begin{array}{ll} A(3 - 3) + B(3 + 3) = 6 & A(-3 - 3) + B(-3 + 3) = 6 \\ 6B = 6 & -6A = 6 \\ B = 1 & A = -1 \end{array}$$

Now substitute these values of A and B back into the rational expressions:

$$\frac{6}{(x + 3)(x - 3)} = -\frac{1}{x + 3} + \frac{1}{x - 3}$$

This sum of two rational expressions is a whole lot friendlier to integrate than what you started with. Use the Sum Rule followed by a simple variable substitution (see Chapter 5):

$$\begin{aligned} & \int \left(-\frac{1}{x+3} + \frac{1}{x-3} \right) dx \\ &= -\int \frac{1}{x+3} dx + \int \frac{1}{x-3} dx \\ &= -\ln |x+3| + \ln |x-3| + C \end{aligned}$$

As with regular fractions, you can't always break rational expressions apart in this fashion. But in four distinct cases, which I discuss in the next section, you can use this technique to integrate complicated rational functions.

Solving Integrals by Using Partial Fractions

In the last section, I show you how to use partial fractions to split a complicated rational function into several smaller and more-manageable functions. Although this technique will certainly amaze your friends, you may be wondering why it's worth learning.

The payoff comes when you start integrating. Lots of times, you can integrate a big rational function by breaking it into the sum of several bite-sized chunks. Here's a bird's-eye view of how to use partial fractions to integrate a rational expression:

- 1. Set up the rational expression as a sum of partial fractions with unknowns (A , B , C , and so forth) in the numerators.**

I call these *unknowns* rather than variables to distinguish them from x , which remains a variable for the whole problem.

- 2. Find the values of all the unknowns and plug them into the partial fractions.**
- 3. Integrate the partial fractions separately by whatever method works.**

In this section, I focus on these three steps. I show you how to turn a complicated rational function into a sum of simpler rational functions and how to replace unknowns (such as A , B , C , and so on) with numbers. Finally, I give you a few important techniques for integrating the types of simpler rational functions that you often see when you use partial fractions.

Setting up partial fractions case by case

Setting up a sum of partial fractions isn't difficult, but there are four distinct cases to watch out for. Each case results in a different setup — some easier than others.



Try to get familiar with these four cases, because I use them throughout this chapter. Your first step in any problem that involves partial fractions is to recognize which case you're dealing with so that you can solve the problem.

Each of these cases is listed in Table 8-1.

Case	Example	As Partial Fractions
Case #1: Distinct linear factors	$\frac{x}{(x+4)(x-7)}$	$\frac{A}{x+4} + \frac{B}{x-7}$
Case #2: Distinct irreducible quadratic factors	$\frac{8}{(x^2+3)(x^2+9)}$	$\frac{A+Bx}{(x-\sqrt{3})(x+\sqrt{3})} + \frac{C+Dx}{x^2+9}$
Case #3: Repeated linear factors	$\frac{2x+2}{(x+5)^2}$	$\frac{A}{x+5} + \frac{B}{(x+5)^2}$
Case #4: Repeated quadratic factors	$\frac{x^2-2}{(x^2+6)^2}$	$\frac{A+Bx}{x^2+6} + \frac{C+Dx}{(x^2+6)^2}$

Case #1: Distinct linear factors

The simplest case in which partial fractions are helpful is when the denominator is the product of *distinct linear factors* — that is, linear factors that are nonrepeating.



For each distinct linear factor in the denominator, add a partial fraction of the following form:

$$\frac{A}{\text{linear factor}}$$

For example, suppose that you want to integrate the following rational expression:

$$\frac{1}{x(x+2)(x-5)}$$

The denominator is the product of three distinct linear factors — x , $(x + 2)$, and $(x - 5)$ — so it's equal to the sum of three fractions with these factors as denominators:

$$= \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-5}$$



The number of distinct linear factors in the denominator of the original expression determines the number of partial fractions. In this example, the presence of three factors in the denominator of the original expression yields three partial fractions.

Case #2: Distinct quadratic factors

Another not-so-bad case where you can use partial fractions is when the denominator is the product of *distinct quadratic factors* — that is, quadratic factors that are nonrepeating.



For each distinct quadratic factor in the denominator, add a partial fraction of the following form:

$$\frac{A + Bx}{\text{quadratic factor}}$$

For example, suppose that you want to integrate this function:

$$\frac{5x - 6}{(x - 2)(x^2 + 3)}$$

The first factor in the denominator is linear, but the second is quadratic and can't be decomposed to linear factors. So, set up your partial fractions as follows:

$$= \frac{A}{x-2} + \frac{Bx+C}{x^2+3}$$



As with distinct linear factors, the number of distinct quadratic factors in the denominator tells you how many partial fractions you get. So in this example, two factors in the denominator yield two partial fractions.

Case #3: Repeated linear factors

Repeated linear factors are more difficult to work with because each factor requires more than one partial fraction.



For each squared linear factor in the denominator, add *two* partial fractions in the following form:

$$\frac{A}{\text{linear factor}} + \frac{B}{(\text{linear factor})^2}$$

For each quadratic factor in the denominator that's raised to the third power, add *three* partial fractions in the following form:

$$\frac{A}{\text{linear factor}} + \frac{B}{(\text{linear factor})^2} + \frac{C}{(\text{linear factor})^3}$$

Generally speaking, when a linear factor is raised to the n th power, add n partial fractions. For example, suppose that you want to integrate the following expression:

$$\frac{x^2 - 3}{(x + 5)(x - 1)^3}$$

This expression contains all linear factors, but one of these factors ($x + 5$) is nonrepeating and the other ($x - 1$) is raised to the third power. Set up your partial fractions this way:

$$= \frac{A}{x + 5} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3}$$

As you can see, I add one partial fraction to account for the nonrepeating factor and three to account for the repeating factor.

Case #4: Repeated quadratic factors

Your worst nightmare when it comes to partial fractions is when the denominator includes repeated quadratic factors.



For each squared quadratic factor in the denominator, add *two* partial fractions in the following form:

$$\frac{Ax + B}{\text{quadratic factor}} + \frac{Cx + D}{(\text{quadratic factor})^2}$$

For each quadratic factor in the denominator that's raised to the third power, add *three* partial fractions in the following form:

$$\frac{Ax + B}{\text{quadratic factor}} + \frac{Cx + D}{(\text{quadratic factor})^2} + \frac{Ex + F}{(\text{quadratic factor})^3}$$

Generally speaking, when a quadratic factor is raised to the n th power, add n partial fractions. For example:

$$\frac{7 + x}{(x - 8)(x^2 + x + 1)(x^2 + 3)^2}$$

This denominator has one nonrepeating linear factor ($x - 8$), one nonrepeating quadratic factor ($x^2 + x + 1$), and one quadratic expression that's squared ($x^2 + 3$). Here's how you set up the partial fractions:

$$= \frac{A}{x - 8} + x^2 + x - 1 = \left(x - \left(-1 + \sqrt{\frac{5}{2}} \right) \right) \cdot \left(x - \left(-1 - \sqrt{\frac{5}{2}} \right) \right) + \frac{D + Ex}{x^2 + 3} + \frac{F + Gx}{(x^2 + 3)^2}$$

This time, I added one partial fraction for each of the nonrepeating factors and two partial fractions for the squared factor.

Beyond the four cases: Knowing how to set up any partial fraction

At the outset, I have some great news: You'll probably never have to set up a partial fraction any more complex than the one that I show you in the previous section. So relax.

I'm aware that some students like to get this stuff on a case-by-case basis, so that's why I introduce it that way. However, other students prefer to be shown an overall pattern, so they can get the Zen math experience. If this is your path, read on. If not, feel free to skip ahead.

You can break *any* rational function into a sum of partial fractions. You just need to understand the pattern for repeated higher-degree polynomial factors in the denominator. This pattern is simplest to understand with an example. Suppose that you're working with the following rational function:

$$\frac{5x + 1}{(7x^4 + 1)^5 (x + 2)^2 (x^2 + 1)^2}$$

In this factor, the denominator includes a problematic factor that's a *fourth-degree polynomial* raised to the *fifth power*. You can't decompose this factor further, so the function falls outside the four cases I outline earlier in this chapter. Here's how you break this rational function into partial fractions:

$$\begin{aligned} &= \frac{Ax^3 + Bx^2 + Cx + D}{7x^4 + 1} + \\ &\frac{Ex^3 + Fx^2 + Gx + H}{(7x^4 + 1)^2} + \\ &\frac{Ix^3 + Jx^2 + Kx + L}{(7x^4 + 1)^3} + \\ &\frac{Mx^3 + Nx^2 + Ox + P}{(7x^4 + 1)^4} + \\ &\frac{Qx^3 + Rx^2 + Sx + T}{(7x^4 + 1)^5} + \\ &\frac{U}{x + 2} + \frac{V}{(x + 2)^2} + \\ &\frac{Wx + X}{(x^2 + 1)} + \frac{Yx + Z}{(x^2 + 1)^2} \end{aligned}$$

As you can see, I completely run out of capital letters. As you can also see, the problematic factor spawns *five partial fractions* — that is, the same number as the power it's raised to. Furthermore:

- ✓ The numerator of each of these fractions is a polynomial of *one degree less* than the denominator.
- ✓ The denominator of each of these fractions is a carbon copy of the original denominator, but in each case raised to a different power up to and including the original.

The remaining two factors in the denominator — a repeated linear (Case #3) and a repeated quadratic (Case #4) — give you the remaining four fractions, which look tiny and simple by comparison.

Clear as mud? Spend a little time with this example and the pattern should become clearer. Notice, too, that the four cases that I outline earlier in this chapter all follow this same general pattern.

You'll probably never have to work with anything as complicated as this — let alone try to integrate it! — but when you understand the pattern, you can break any rational function into partial fractions without worrying which case it is.

Knowing the ABCs of finding unknowns

You have two ways to find the unknowns in a sum of partial fractions. The easy and quick way is by using the roots of polynomials. Unfortunately, this method doesn't always find all the unknowns in a problem, though it often finds a few of them. The second way is to set up a system of equations.

Rooting out values with roots

When a sum of partial fractions has linear factors (either distinct or repeated), you can use the roots of these linear factors to find the values of unknowns. For example, in the earlier section "Case #1: Distinct linear factors," I set up the following equation:

$$\frac{1}{x(x+2)(x-5)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-5}$$

To find the values of the unknowns A , B , and C , first get a common denominator on the right side of this equation (the same denominator that's on the left side):

$$\frac{1}{x(x+2)(x-5)} = \frac{A(x+2)(x-5) + Bx(x-5) + Cx(x+2)}{x(x+2)(x-5)}$$

Now, multiply both sides by this denominator:

$$1 = A(x+2)(x-5) + Bx(x-5) + Cx(x+2)$$

To find the values of A , B , and C , substitute the roots of the three factors (0, -2, and 5):

$$1 = A(2)(-5)$$

$$1 = B(-2)(-2-5)$$

$$1 = C(5)(5+2)$$

$$A = -\frac{1}{10}$$

$$B = \frac{1}{14}$$

$$C = \frac{1}{35}$$

Plugging these values back into the original integral gives you:

$$-\frac{1}{10x} + \frac{1}{14(x+2)} + \frac{1}{35(x-5)}$$

This expression is equivalent to what you started with, but it's much easier to integrate. To do so, use the Sum Rule to break it into three integrals, the Constant Multiple Rule to move fractional coefficients outside each integral, and variable substitution (see Chapter 5) to do the integration. Here's the answer so that you can try it out:

$$\begin{aligned} \int \left[-\frac{1}{10x} + \frac{1}{14(x+2)} + \frac{1}{35(x-5)} \right] dx \\ = -\frac{1}{10} \ln x + \frac{1}{14} \ln(x+2) + \frac{1}{35} \ln(x-5) + K \end{aligned}$$



In this answer, I use K rather than C to represent the constant of integration to avoid confusion, because I already use C in the earlier partial fractions.

Working systematically with a system of equations

Setting up a system of equations is an alternative method for finding the value of unknowns when you're working with partial fractions. It's not as simple as plugging in the roots of factors (which I show you in the last section), but it's your only option when the root of a quadratic factor is imaginary.

To illustrate this method and why you need it, I use the problem that I set up in "Case #2: Distinct quadratic factors":

$$\frac{5x-6}{(x-2)(x^2+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+3}$$

To start out, see how far you can get by plugging in the roots of equations. As I show you in "Rooting out values with roots," begin by getting a common denominator on the right side of the equation:

$$\frac{5x-6}{(x-2)(x^2+3)} = \frac{(A)(x^2+3) + (Bx+C)(x-2)}{(x-2)(x^2+3)}$$

Now, multiply the whole equation by the denominator:

$$5x - 6 = (A)(x^2 + 3) + (Bx + C)(x - 2)$$

The root of $x - 2$ is 2, so let $x = 2$ and see what you get:

$$5(2) - 6 = A(2^2 + 3)$$

$$A = \frac{4}{7}$$

Now, you can substitute $\frac{4}{7}$ for A :

$$5x - 6 = \frac{4}{7(x^2 + 3)} + (Bx + C)(x - 2)$$

Unfortunately, $x^2 + 3$ has no root in the real numbers, so you need a different approach. First, get rid of the parentheses on the right side of the equation:

$$5x - 6 = \frac{4}{7}x^2 + \frac{12}{7} + Bx^2 - 2Bx + Cx - 2C$$

Next, combine similar terms (using x as the variable by which you judge similarity). This is just algebra, so I skip a few steps here:

$$x^2\left(\frac{4}{7} + B\right) + x(-2B + C - 5) + \left(\frac{12}{7} - 2C + 6\right) = 0$$

Because this equation works for *all* values of x , I now take what appears to be a questionable step, breaking this equation into three separate equations as follows:

$$\frac{4}{7} + B = 0$$

$$-2B + C - 5 = 0$$

$$\frac{12}{7} - 2C + 6 = 0$$

At this point, a little algebra tells you that $B = -\frac{4}{7}$ and $C = \frac{27}{7}$. So you can substitute the values of A , B , and C back into the partial fractions:

$$\frac{5x - 6}{(x - 2)(x^2 + 3)} = \frac{4}{7(x - 2)} + \frac{-\frac{4}{7}x + \frac{27}{7}}{x^2 + 3}$$

You can simplify the second fraction a bit:

$$\frac{4}{7(x - 2)} + \frac{-4x + 27}{7(x^2 + 3)}$$

Integrating partial fractions

After you express a hairy rational expression as the sum of partial fractions, integrating becomes a lot easier. Generally speaking, here's the system:

1. **Split all rational terms with numerators of the form $Ax + B$ into two terms.**
2. **Use the Sum Rule to split the entire integral into many smaller integrals.**
3. **Use the Constant Multiple Rule to move coefficients outside each integral.**
4. **Evaluate each integral by whatever method works.**

Linear factors: Cases #1 and #3

When you start out with a linear factor — whether distinct (Case #1) or repeated (Case #3) — using partial fractions leaves you with an integral in the following form:

$$\int \frac{1}{(ax+b)^n} dx$$

Integrate all these cases by using the variable substitution $u = ax + b$ so that $du = a dx$ and $\frac{du}{a} = dx$. This substitution results in the following integral:

$$= \frac{1}{a} \int \frac{1}{u^n} du$$

Here are a few examples:

$$\int \frac{1}{3x+5} dx = \frac{1}{3} \ln|3x+5| + C$$

$$\int \frac{1}{(6x-1)^2} dx = -\frac{1}{6(6x-1)} + C$$

$$\int \frac{1}{(x+9)^3} dx = -\frac{1}{2(x+9)^2} + C$$

Quadratic factors of the form $(ax^2 + C)$: Cases #2 and #4

When you start out with a quadratic factor of the form $(ax^2 + C)$ — whether distinct (Case #2) or repeated (Case #4) — using partial fractions results in the following two integrals:

$$\int \frac{x}{(ax^2 + C)^n} dx$$

$$\int \frac{1}{(ax^2 + C)^n} dx$$

Integrate the first by using the variable substitution $u = ax^2 + C$ so that $du = ax \, dx$ and $\frac{du}{a} = x \, dx$. This substitution results in the following integral:

$$= \frac{1}{a} \int \frac{1}{u^n} du$$

This is the same integral that arises in the linear case that I describe in the previous section. Here are some examples:

$$\begin{aligned} \int \frac{x}{7x^2 + 1} dx &= \frac{1}{14} \ln|7x^2 + 1| + C \\ \int \frac{x}{(x^2 + 4)^2} dx &= -\frac{2}{(x^2 + 4)} + C \\ \int \frac{x}{(8x^2 - 2)^3} dx &= \frac{-1}{32(8x^2 - 2)^2} + C \end{aligned}$$

To evaluate the second integral, use the following formula:

$$\int \frac{1}{x^2 + n^2} dx = \frac{1}{n} \arctan \frac{x}{n} + C$$

Quadratic factors of the form $(ax^2 + bx + C)$: Cases #2 and #4

Most math teachers have at least a shred of mercy in their hearts, so they don't tend to give you problems that include this most difficult case. When you start out with a quadratic factor of the form $(ax^2 + bx + C)$ — whether distinct (Case #2) or repeated (Case #4) — using partial fractions results in the following integral:

$$\int \frac{hx + k}{(ax^2 + bx + C)^n} dx$$

I know, I know — that's way too many letters and not nearly enough numbers. Here's an example:

$$\int \frac{x - 5}{x^2 + 6x + 13} dx$$

This is about the hairiest integral you're ever going to see at the far end of a partial fraction. To evaluate it, you want to use the variable substitution $u = x^2 + 6x + 13$ so that $du = (2x + 6) \, dx$. If the numerator were $2x + 6$, you'd be in great shape. So you need to tweak the numerator a bit. First multiply it by 2 and divide the whole integral by 2:

$$= \frac{1}{2} \int \frac{2x - 10}{x^2 + 6x + 13} dx$$

Because you multiplied the entire integral by 1, no net change has occurred. Now, add 16 and -16 to the numerator:

$$= \frac{1}{2} \int \frac{2x + 6 - 16}{x^2 + 6x + 13} dx$$

This time, you add 0 to the integral, which doesn't change its value. At this point, you can split the integral in two:

$$= \frac{1}{2} \left[\int \frac{2x + 6}{x^2 + 6x + 13} dx - 16 \int \frac{1}{x^2 + 6x + 13} dx \right]$$

At this point, you can use the desired variable substitution (which I mention a few paragraphs earlier) to change the first integral as follows:

$$\begin{aligned} \int \frac{2x + 6}{x^2 + 6x + 13} dx &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |x^2 + 6x + 13| + C \end{aligned}$$

To solve the second integral, complete the square in the denominator: Divide the b term (6) by 2 and square it, and then represent the C term (13) as the sum of this and whatever's left:

$$-16 \int \frac{1}{x^2 + 6x + 9 + 4} dx$$

Now, split the denominator into two squares:

$$= -16 \int \frac{1}{(x + 3)^2 + 2^2} dx$$

To evaluate this integral, use the same formula that I show you in the previous section:

$$\int \frac{1}{x^2 + n^2} dx = \frac{1}{n} \arctan \frac{x}{n} + C$$

So here's the final answer for the second integral:

$$-8 \arctan \frac{x + 3}{2} + C$$

Therefore, piece together the complete answer as follows:

$$\begin{aligned} &\int \frac{x - 5}{x^2 + 6x + 13} dx \\ &= \frac{1}{2} \left[\ln |x^2 + 6x + 13| - 8 \arctan \frac{x + 3}{2} \right] + C \\ &= \frac{1}{2} \ln |x^2 + 6x + 13| - 4 \arctan \frac{x + 3}{2} + C \end{aligned}$$

Integrating Improper Rationals

Integration by partial fractions works only with *proper rational expressions*, but not with *improper rational expressions*. In this section, I show you how to tell these two beasts apart. Then I show you how to use polynomial division to turn improper rationals into more acceptable forms. Finally, I walk you through an example in which you integrate an improper rational expression by using everything in this chapter.

Distinguishing proper and improper rational expressions

Telling a proper fraction from an improper one is easy: A fraction $\frac{a}{b}$ is proper if the numerator (disregarding sign) is *less* than the denominator, and improper otherwise.

With rational expressions, the idea is similar, but instead of comparing the value of the numerator and denominator, you compare their *degrees*. The degree of a polynomial is its highest power of x (flip to Chapter 2 for a refresher on polynomials).



A rational expression is proper if the degree of the numerator is less than the degree of the denominator, and improper otherwise.

For example, look at these three rational expressions:

$$\frac{x^2 + 2}{x^3}$$

$$\frac{x^5}{3x^2 - 1}$$

$$\frac{-5x^4}{3x^4 - 2}$$

In the first example, the numerator is a second-degree polynomial and the denominator is a third-degree polynomial, so the rational is *proper*. In the second example, the numerator is a fifth-degree polynomial and the denominator is a second-degree polynomial, so the expression is *improper*. In the third example, the numerator and denominator are both fourth-degree polynomials, so the rational function is *improper*.

Recalling polynomial division

Most math students learn polynomial division in Algebra II, demonstrate that they know how to do it on their final exam, and then promptly forget it. And, happily, they never need it again — except to pass the time at *extremely* dull parties — until Calculus II.

It's time to take polynomial division out of mothballs. In this section, I show you everything you forgot to remember about polynomial division, both with and without a remainder.

Polynomial division without a remainder

When you multiply two polynomials, you always get another polynomial. For example:

$$(x^3 + 3)(x^2 - x) = x^5 - x^4 + 3x^2 - 3x$$

Because division is the inverse of multiplication, the following equation makes intuitive sense:

$$\frac{x^5 - x^4 + 3x^2 - 3x}{x^3 + 3} = (x^2 - x)$$

Polynomial division is a reliable method for dividing one polynomial by another. It's similar to long division, so you probably won't have too much difficulty understanding it even if you've never seen it.

The best way to show you how to do polynomial division is with an example. Start with the example I've already outlined. Suppose that you want to divide $x^5 - x^4 + 3x^2 - 3x$ by $x^3 + 3$. Begin by setting up the problem as a typical long division problem (notice that I fill with zeros for the x^3 and constant terms):

$$x^3 + 3 \overline{) x^5 - x^4 + 0x^3 + 3x^2 - 3x + 0}$$

Start by focusing on the highest degree exponent in both the divisor (x^3) and dividend (x^5). Ask how many times x^3 goes into x^5 — that is, $x^5 \div x^3 = ?$ Place the answer in the quotient, and then multiply the result by the divisor as you would with long division:

$$\begin{array}{r} x^2 \\ x^3 + 3 \overline{) x^5 - x^4 + 0x^3 + 3x^2 - 3x + 0} \\ \underline{-(x^5 \qquad \qquad + 3x^2)} \end{array}$$

As you can see, I multiply x^2 by x^3 to get the result of $x^5 + 3x^2$, aligning this result to keep terms of the same degree in similar columns. Next, subtract and bring down the next term, just as you would with long division:

$$\begin{array}{r} x^3 + 3 \overline{) x^5 - x^4 + 0x^3 + 3x^2 - 3x + 0} \\ -(x^5 \qquad \qquad + 3x^2) \\ \hline -x^4 \qquad \qquad \qquad - 3x \end{array}$$

Now, the cycle is complete, and you ask how many times x^3 goes into $-x^4$ — that is, $-x^4 \div x^3 = ?$ Place the answer in the quotient, and multiply the result by the divisor:

$$\begin{array}{r} x^3 + 3 \overline{) x^5 - x^4 + 0x^3 + 3x^2 - 3x + 0} \\ -(x^5 \qquad \qquad + 3x^2) \\ \hline -x^4 \qquad \qquad \qquad - 3x \\ -(-x^4 \qquad \qquad - 3x) \\ \hline \end{array}$$

In this case, the subtraction that results works out evenly. Even if you bring down the final zero, you have nothing left to divide, which shows the following equality:

$$\frac{x^5 - x^4 + 3x^2 - 3x}{x^3 + 3} = x^2 - x$$

Polynomial division with a remainder

Because polynomial division looks so much like long division, it makes sense that polynomial division should, at times, leave a remainder. For example, suppose that you want to divide $x^4 - 2x^3 + 5x$ by $2x^2 - 6$:

$$2x^2 - 6 \overline{) x^4 - 2x^3 + 0x^2 + 5x + 0}$$

This time, I fill in two zero coefficients as needed. To begin, divide x^4 by $2x^2$, multiply through, and subtract:

$$\begin{array}{r} \frac{1}{2}x^2 \\ 2x^2 - 6 \overline{) x^4 - 2x^3 + 0x^2 + 5x + 0} \\ -(x^4 \qquad \qquad - 3x^2) \\ \hline -2x^3 + 3x^2 \end{array}$$



Don't let the fractional coefficient deter you. Sometimes polynomial division results in fractional coefficients.

Now bring down the next term ($5x$) to begin another cycle. Then, divide $-2x^3$ by $2x^2$, multiply through, and subtract:

$$\begin{array}{r}
 \frac{1}{2}x^2 - x \\
 2x^2 - 6 \overline{) x^4 - 2x^3 + 0x^2 + 5x + 0} \\
 \underline{-(x^4 \quad - 3x^2)} \\
 -2x^3 + 3x^2 + 5x \\
 \underline{-(-2x^3 \quad + 6x)} \\
 3x^2 - x
 \end{array}$$

Again, bring down the next term (0) and begin another cycle by dividing $3x^2$ by $2x^2$:

$$\begin{array}{r}
 \frac{1}{2}x^2 - x + \frac{3}{2} \\
 2x^2 - 6 \overline{) x^4 - 2x^3 + 0x^2 + 5x + 0} \\
 \underline{-(x^4 \quad - 3x^2)} \\
 -2x^3 + 3x^2 + 5x \\
 \underline{-(-2x^3 \quad + 6x)} \\
 3x^2 - x + 0 \\
 \underline{-(3x^2 \quad - 9)} \\
 -x + 9
 \end{array}$$

As with long division, the remainder indicates a fractional amount left over: the remainder divided by the divisor. So, when you have a remainder in polynomial division, you write the answer by using the following formula:

$$\text{Polynomial} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$



If you get confused deciding how to write out the answer, think of it as a mixed number. For example, $7 \div 3 = 2$ with a remainder of 1, which you write as $2\frac{1}{3}$.

So, the polynomial division in this case provides the following equality:

$$\frac{x^4 - 2x^3 + 5x}{2x^2 - 6} = \frac{1}{2}x^2 - x + \frac{3}{2} + \frac{-x + 9}{2x^2 - 6}$$

Although this result may look more complicated than the fraction you started with, you have made progress: You turned an improper rational expression (where the degree of the numerator is greater than the degree of the denominator) into a sum that includes a proper rational expression. This is similar to the practice in arithmetic of turning an improper fraction into a mixed number.

Trying out an example

In this section, I walk you through an example that takes you through just about everything in this chapter. Suppose that you want to integrate the following rational function:

$$\frac{x^4 - x^3 - 5x + 4}{(x - 2)(x^2 + 3)} dx$$

This looks like a good candidate for partial fractions, as I show you earlier in this chapter in “Case #2: Distinct quadratic factors.” But before you can express it as partial fractions, you need to determine whether it’s proper or improper. The degree of the numerator is 4 and (because the denominator is the product of a linear and a quadratic) the degree of the entire denominator is 3. Thus, this is an improper polynomial fraction (see “Distinguishing proper and improper rational expressions” earlier in this chapter), so you can’t integrate by parts.

However, you can use polynomial division to turn this improper polynomial fraction into an expression that includes a proper polynomial fraction (I omit these steps here, but I show you how earlier in this chapter in “Recalling polynomial division.”):

$$\frac{x^4 - x^3 - 5x + 4}{(x - 2)(x^2 + 3)} = x + 1 + \frac{-x^2 - 2x + 10}{(x - 2)(x^2 + 3)}$$

As you can see, the first two terms of this expression are simple to integrate (don’t forget about them!). To set up the remaining term for integration, use partial fractions:

$$\frac{-x^2 - 2x + 10}{(x - 2)(x^2 + 3)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 3}$$

Get a common denominator on the right side of the equation:

$$\frac{-x^2 - 2x + 10}{(x - 2)(x^2 + 3)} = \frac{A(x^2 + 3) + (Bx + C)(x - 2)}{(x - 2)(x^2 + 3)}$$

Now multiply both sides of the equation by this denominator:

$$-x^2 - 2x + 10 = A(x^2 + 3) + (Bx + C)(x - 2)$$

Notice that $(x - 2)$ is a linear factor, so you can use the root of this factor to find the value of A . To find this value, let $x = 2$ and solve for A :

$$-(2^2) - 2(2) + 10 = A(2^2 + 3) + (B2 + C)(2 - 2)$$

$$2 = 7A$$

$$A = \frac{2}{7}$$

Substitute this value into the equation:

$$-x^2 - 2x + 10 = \frac{2}{7}(x^2 + 3) + (Bx + C)(x - 2)$$

At this point, to find the values of B and C , you need to split the equation into a system of two equations (as I show you earlier in “Working systematically with a system of equations”):

$$\begin{aligned} -x^2 - 2x + 10 &= \frac{2}{7}x^2 + \frac{6}{7} + Bx^2 + Cx - 2Bx - 2C \\ \left(\frac{2}{7} + B + 1\right)x^2 + (-2B + C + 2)x + \left(\frac{6}{7} - 2C - 10\right) &= 0 \end{aligned}$$

This splits into three equations:

$$\frac{2}{7} + B + 1 = 0$$

$$-2B + C + 2 = 0$$

$$\frac{6}{7} - 2C - 10 = 0$$

The first and the third equations show you that $B = -\frac{9}{7}$ and $C = -\frac{32}{7}$. Now you can plug the values of A , B , and C back into the sum of partial fractions:

$$\frac{2}{7(x-2)} + \frac{-9x-32}{7(x^2+3)}$$

Make sure that you remember to add in the two terms $(x + 1)$ that you left behind just after you finished your polynomial division:

$$\int \frac{x^4 - x^3 - 5x + 4}{(x-2)(x^2+3)} dx = \int \left[x + 1 + \frac{2}{7(x-2)} + \frac{9x-32}{7(x^2+3)} \right] dx$$

Thus, you can rewrite the original integral as the sum of five separate integrals:

$$\int x dx + \int dx + \frac{2}{7} \int \frac{1}{x-2} dx - \frac{9}{7} \int \frac{x}{x^2+3} dx - \frac{32}{7} \int \frac{1}{x^2+3} dx$$

You can solve the first two of these integrals by looking at them, and the next two by variable substitution (see Chapter 5). The last is done by using the following rule:

$$\int \frac{1}{x^2+n^2} dx = \frac{1}{n} \arctan \frac{x}{n} + C$$

Here's the solution so that you can work the last steps yourself:

$$\frac{1}{2}x^2 + x + \frac{2}{7} \ln|x-2| - \frac{9}{14} \ln|x^2+3| - \frac{32}{7\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C$$

